Arithmetical Foundations

Recursion. Evaluation. Consistency

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Preface

Recursive maps—nowadays called *primitive recursive maps*—have been introduced by GÖDEL in his 1931 article for the arithmetisation—*gödelisation*—of Metamathematics.

For construction of his *undecidable formula* he introduces a non-constructive—non PR—predicate *beweisbar,* non-prov*able.*

If we remain within the area of (categorical) Free-Variables Theory **PR** of Primitive Recursion—or appropriate extensions—we have the chance to avoid the two GÖDEL's Incompleteness Theorems, stated for *Principia Mathematica und verwandte Systeme* such as in particular Zermelo-Fraenkel **set** theory **ZF** : we keep outside of our theories in particular (existential) quantor '$\exists$'.

On the basis of Primitive Recursion we introduce μ-recursive maps as *partial PR maps* and special *terminating* general recursive maps—terminating by additional axiom, (π)—as *(descending) complexity controlled* iterations, *complexity* taking values within appropriate (countable) Ordinal $\mathbb{N}[\omega] \equiv \omega^\omega$ of polynomials in one indeterminate.

Code evaluation then is given in terms of such an iteration. In the light of evaluation we discuss consistency provability and soundnesss for theories of Primitive Recursion as well as for classical, quantified arithmetical and **set** theories, with unexpected results:

We get *decidability* of all PR predicates by Theory $\pi\mathbf{R}$ obtained from **PR** by adding (already mentioned) axiom (π) of non-infinite *descent,* and from that in particular decidability of *consistency provability* and hence $\pi\mathbf{R}$-derivability of $\pi\mathbf{R}$'s own (free-variable) Consistency formula $\mathrm{Con}_{\pi\mathbf{R}}$—$\pi\mathbf{R}$ **assumed** to be "π-consistent."

By the same token is obtained ω-*inconsistency* of classical **set** theories, using (contraposition of) Gödel's *2nd Incompleteness Theorem.*

For the time being, present "monograph" needs discussion. I look forward to your comments, critiques, reports on errors, suggestions, hints to (your work and) the work of other people.

Berlin, August 2012 M. Pfender.

P. S. I am obviously not an English native speaker. As Joseph Helfer puts it, my mathematical thinking and speach is somewhat special, it is *Germish.*

Contents

Introduction

We attempt here to fix *Constructive Foundations* for Arithmetic on a *map* theoretical, *algorithmical* level.

In contrast to *elementhood* and *quantification* based traditional Foundations such as Principia Mathematica **PM** or Zermelo-Fraenkel set Theory **ZF**, "our" *Fundamental Primitive Recursive Theory* **PR** has as its "undefined" terms just terms for Objects and maps, and on that language level it is *variable free,* and it is free from (formal) quantification.

This theory **PR** is a formal, *combinatorial category* with Cartesian i. e. universal *Product*—a *Cartesian* Theory—with Natural Numbers Object $\mathbb{N}$. Theory **PR** is equipped with an NNO in the sense that **PR** is a (combinatorially generated) formal Cartesian Theory admitting *iteration of endo maps* and the *full schema of Primitive Recursion,* in categorical terms introduced by FREYD 1972.

We remain on the purely **syntactical** level of this categorical theory, and later (categorical) **extensions:** *no formal Semantics* necessary into an outside, non-combinatorial world, cf. Hilbert's formalistic programme.

We develop fundamental (categorical) PR Theory **PR** from endo iteration schema (§) of Eilenberg & Elgot 1970, and—again taken as **axiom**—Freyd's *Uniqueness of initialised iterated endo map* to give derivation of "his" full schema of Primitive Recursion, including in particular *uniqueness* of PR map defined by that schema, out of *initialisation map* and *step map.*

We then introduce—into our *variable-free* setting—*Free Variables,* which come in by **interpretation** of these variables as (another) *names* for identities resp. (possibly nested) left/right (Cartesian) *projections,* and the combinatorial rules for their "legitimate" use: see section on *Introduction of Free Variables.*

As a consequence, we then have in present context '*Free Variable*' as a *defined* notion, and we have Object and map *constants* such as *terminal Object,* NNO etc. as well as meta Free Variables for Objects and for maps.

Fundamental Arithmetic is further developed along Goodstein's FV Arithmetic whose *uniqueness rules* are derived as Theorems of (categorical) Theory **PR**—with its "eliminable" notion of a *Free Variable.*

This gives—expected—*Structure Theorem for Algebra and Order* on NNO $\mathbb{N}$, and, "on the way", via Goodstein's *truncated subtraction,* and "his" *commutativity of maximum function,* the

Equality Definability Theorem: If predicative equality of two PR maps is derivably true, then map equality between these maps is derivable.

It follows a section on derivation of Peano-axioms as Theorems.

Second "half" of chapter 1 brings into the game an (embedding) Theory Extension of **PR** by *abstraction* of *predicates* into "virtual" (new) *Objects.* This enrichment makes emerging *basic* Theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ more comfortable, in direction to **set** theories.

We exhibit within Theory **PRa** a *Universal Object,* $\mathbb{X}$, and construct by means of that Object another presentation $\mathbf{PRa}\mathbb{X} \cong \mathbf{PRa}$. Theory $\mathbf{PRa}\mathbb{X}$—central *basic Cartesian PR Theory* for all what follows—introduces partiality of *defined* partial maps (with *partiality* defined by a PR predicate), by *case distinction* on the *Domain of defined arguments* within a surounding **PR** Domain Object.

Chapter 2 introduces *partial* (PR) maps as map pairs consisting of a *domain-of-definition enumeration* on one hand and on the other hand of a *rule* to throw enumeration index of *defined argument* into the *value* of that argument.

We define *equality* of partial maps by availability of extension maps between the enumeration Domains of the two partial maps under consideration, in both directions.

These partial maps form a Primitive Recursive diagonal-monoidal half-Cartesian Theory $\mathbf{P\widehat{R}a}$ which contains Theories $\mathbf{PRa}\mathbb{X} \cong \mathbf{PRa}$ embedded as theory of this type, composition defined via composition of pullbacks.

This is the main assertion of our **Structure Theorem.** Theory extension by partiality is a *Closure* operator: *partial* partial maps are just partial maps.

The last two sections of chapter 2 introduce μ-recursive maps and <u>while</u> loops map theoretically as partial PR maps and show equivalence of all three concepts, cf. Church's Thesis.

In chapter 3 on *Evaluation and Soundness,* we strengthen PR Theory $\mathbf{PRa}\mathbb{X}$ into *Descent Theory* $\pi\mathbf{R}$, by an axiom of *non-infinite it-*

erative descent with order values in polynomial semiring $\mathbb{N}[\omega] \subset \mathbb{N}^*$ ordered lexicographically.

This theory is shown to derive the—free variable PR—consistency formula for PR Theories $\mathbf{PRa}\mathbb{X}$ (and $\mathbf{PR}$). The proof relies on constructive, complexity controlled code evaluation, which is extended to evaluation of argumented deduction trees:

E & S Theorem on *Termination Conditioned Soundness of* $\mathbf{PRa} \cong \mathbf{PRa}\mathbb{X}$ *within Theory* $\pi\mathbf{R}$ *taken as frame.*

Consequence is consistency formula *decidability* and then—ω-consistency resp. "π-consistency" ***assumed*** (π-consistency being an adaptation of ω-consistency to the case of quantifier-free PR theories)—consistency *provability* for **set** theories $\mathbf{T}$ as well as for *Descent* Theory $\pi\mathbf{R}$.

By (contraposition of) Gödel's 2nd Incompleteness Theorem this decidability for **set** theories $\mathbf{T}$ entails ω-inconsistency of $\mathbf{PM}, \mathbf{ZF}$ and of other **set** theories. *Self-consistency* of—recursive—Descent theory $\pi\mathbf{R}$ comes somewhat unexpected, but formally it is not excluded by Gödel's Incompleteness Theorems.

Chapter 1

Primitive Recursion

1.1 Fundamental Theory PR of Primitive Recursion

We fix **terms** and **axioms** for (a first, "short") *presentation* of the—*fundamental*—categorical (formally variable-free) Cartesian Theory **PR** of Primitive Recursion, as follows:

Object Terms: *basic* Objects—Object terms—of theory **PR** are the *terminal* Object $\mathbb{1}$ and the *Natural Numbers Object* ('NNO') $\mathbb{N}$.

Composed Object terms of **PR** come in as *"Cartesian" Products* $(A \times B)$ of Objects already enumerated. The latter formally:

$$(\mathrm{Obj}_{\mathrm{Cart}}) \quad \frac{A, B \text{ Objects}}{(A \times B) \text{ Object}}$$

[Here outmost brackets may be dropped]
Map Terms: *Basic maps* ("map constants") of theory **PR** are

zero map $0 : \mathbb{1} \to \mathbb{N}$, as well as
successor map $s : \mathbb{N} \to \mathbb{N}$

Structure of PR as a Category:

- generation—enumeration—of *identic maps*

$$\text{(id generation)} \quad \frac{A \text{ an Object}}{\mathrm{id}_A : A \to A \text{ map}}$$

- Composition:

$$(\circ) \quad \frac{f : A \to B,\ \ g : B \to C \text{ maps}}{\begin{array}{l}(g \circ f) : A \to C \text{ map, as diagram:} \\ A \xrightarrow{f} B \xrightarrow{g} C\end{array}}$$

Equality $=^{\mathbf{T}}$ for a map based—categorical—theory **T** is given by external PR enumeration, constructively it is a priori not (PR) decidable, i. e. not a PR (meta) predicate.

Here are—first—the (equational) **axioms** making **PR** into a Category:

- Enumeration of equality for Theory **PR** to be constructed begins with **associativity** of **composition:**

$$(\circ_{\mathrm{ass}}) \quad \frac{f : A \to B,\ g : B \to C,\ h : C \to D \text{ maps}}{h \circ (g \circ f) = (h \circ g) \circ f : A \to D}$$

as well as

- **neutrality** of **identities**

$$(\mathrm{neutr_{id}}) \quad \frac{f : A \to B \text{ map}}{\begin{array}{l}(f \circ \mathrm{id}_A) = f : A \to A \to B \quad \text{and} \\ (\mathrm{id}_B \circ f) = f : A \to B \to B.\end{array}}$$

Building a theory, **equivalence** properties of the notion of equality as well as of **compatibility** with the **defining** meta operations for the theory do in general not come free.

So we are "urged" to state as **axioms** *reflexivity, symmetry* and *transitivity* of map equality $f = g : A \to B$, as follows:

$$(\mathrm{refl}) \quad \frac{f : A \to B \text{ map}}{f = f : A \to B}$$

$$(\text{sym}) \quad \frac{f = g : A \to B \text{ map}}{g = f : A \to B}$$

$$(\text{trans}) \quad \frac{f = g, \;\; g = h : A \to B \text{ maps}}{f = h : A \to B}$$

In the spirit of LEIBNIZ' *Substitutivity of equality* we “have to” state by **axiom** *compatibility* of *composition* with equality, making this equality $f = g : A \to B$ between maps into a congruence, here (first): a congruence with respect to composition:

$$(\circ_=) \quad \frac{f = f' : A \to B, \;\; g = g' : B \to C}{(g \circ f) = (g' \circ f') : A \to B \to C}$$

Because of technical simplicity in (later) code evaluation, we split this axiom (schema) into the following two ones:

$$(\circ_= \text{1st}) \quad \frac{f = f' : A \to B, \;\; g : B \to C}{(g \circ f) = (g \circ f') : A \to B \to C}$$

$$(\circ_= \text{2nd}) \quad \frac{f : A \to B,\ g = g' : B \to C}{(g \circ f) = (g' \circ f) : A \to B \to C}$$

[There is no need for a corresponding axiom for meta operation $A \mapsto \langle \mathrm{id}_A : A \to A \rangle$, since equality on Objects is **discrete,** nothing to preserve. This here as well as for all (categorical) Theories to be discussed in the below]

This terminates the (inference-type) enumeration of **terms** and **axioms** of Theory **PR** as a Category.

Cartesian Map Structure:

- enumeration of *terminal maps*

$$\frac{A \text{ Object}}{\Pi = \Pi_A : A \to \mathbb{1} \text{ map}}$$

[EILENBERG & ELGOT's notation, this *projection* called $! : A \to \mathbb{1}$ by LAWVERE]

- uniqueness **axiom** for terminal map family:

$$(\Pi) \quad \frac{A \text{ Object},\ f : A \to \mathbb{1} \text{ map}}{f = \ \Pi_A : A \to \mathbb{1}}$$

- generation of left and right *projections:*

$$
(\text{proj}) \quad \frac{A,\ B \text{ Objects}}{\begin{array}{l} \ell = \ell_{A,B} : A \times B \to A \ \ \textit{left projection,} \\ r = r_{A,B} : A \times B \to B \ \ \textit{right projection} \end{array}}
$$

- generation of *induced maps* into Products:

$$
(\text{ind}) \quad \frac{f : C \to A,\ g : C \to B \quad \text{maps (same Domain)}}{(f, g) : C \to A \times B \quad \text{map, the } \textit{induced} \text{ by } f \text{ and } g}
$$

- compatibility of induced map formation with equality:

$$
(\text{ind}_=) \quad \frac{f = f' : C \to A,\ g = g' : C \to B \quad \text{maps}}{(f, g) = (f', g') : C \to A \times B}
$$

- Characteristic—Godement—equations

$$
(\text{GODE}_\ell) \quad \frac{f : C \to A,\ g : C \to B}{\ell \circ (f, g) = f : C \to A}
$$

as well as

$$(\mathrm{GODE}_r) \quad \frac{f : C \to A,\ g : C \to B}{r \circ (f, g) = g : C \to B}$$

in (*commutative*) diagram form:

$$\begin{tikzcd} & A \\ C \arrow[ur, bend left, "f"] \arrow[r, "(f{,}g)"] \arrow[dr, bend right, "g"'] & A \times B \arrow[u, "\ell"'] \arrow[d, "r"] \\ & B \end{tikzcd}$$

GODEMENT equations DIAGRAM

- uniqueness of induced map (GODEMENT):

$$(\mathrm{ind!}) \quad \frac{\begin{array}{l} f : C \to A,\ g : C \to B,\ h : C \to A \times B \text{ maps,} \\ \ell \circ h = f : C \to A \text{ and } r \circ h = g : C \to B \end{array}}{h = (f, g) : C \to A \times B}$$

In presence of the other axioms, this *uniqueness of the induced map* is equivalent to the following—purely equational—**axiom** (FOURMAN):

$$\text{(FM)} \quad \frac{h : C \to A \times B}{(\ell \circ h\,,\, r \circ h) = h : C \to A \times B}$$

This even without **axioms** (GODE_ℓ) and (GODE_r) above.

Proof as **Exercise:** Use compatibility of induced with equality.

We will formally **rely** on this FOURMAN's equation as an **axiom.**

This terminates the list of map **terms** and **axioms** making up the Cartesian Structure of Theory concerned, here of Theory **PR**.

An alternative form—many *simpler* **terms** and **axioms**—to present the Cartesian structure, here of Theory **PR**, will be listed below, together with natural Bifunctoriality of Cartesian product.

We now turn to discussion of

Primitive Recursion via Iteration: We introduce *iteration of endo maps*[1] as follows:

$$(\S) \quad \frac{f : A \to A \text{ (endo) map}}{\begin{array}{l} f^\S : A \times \mathbb{N} \to A \textit{ iterated of } f, \text{ to satisfy} \\ f^\S \circ (\mathrm{id}_A, 0\,\Pi_\mathbb{N}) = \mathrm{id}_A : A \to A \text{ (anchor) as well as} \\ f^\S \circ (A \times s) = f \circ f^\S : A \times \mathbb{N} \to A \to A. \text{ (step)} \end{array}}$$

[1] *one-successor* version of EILENBERG & ELGOT 1970

In ***free-variables*** notation:

$$f^\S(a, 0) = a \qquad \text{(anchor)},$$

as well as

$$f^\S(a, s\,n) = f(f^n(a)) \ =_{\text{by def}} \ f(f^\S(a, n)) \qquad \text{(step)},$$

this (equational) **postcedent** in commuting-diagram form, "pentagonal" diagram:

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[dd, "f^\S"] & A \times \mathbb{N} \arrow[dd, "f^\S"] \\
A \arrow[ur, "(\mathrm{id},0\,\Pi)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
 & A \arrow[r, "f"] & A
\end{tikzcd} \qquad \text{(it)}$$

Basic Iteration DIAGRAM

Notation: Here we note, for a map $g : B \to B'$,

$$A \times g \ =_{\text{def}} \ \mathrm{id}_A \times g \ =_{\text{def}} \ (\mathrm{id}_A \circ \ell, g \circ r) : A \times B \to A \times B'.$$

As a first **example** for an iterated endo map take *addition*

$$+ = +(a, n) = (a + n) \ =_{\text{def}} \ s^\S : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

having properties

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[dd, "s^\S", "+"'] & A \times \mathbb{N} \arrow[dd, "s^\S", "+"'] \\
A \arrow[ur, "(\mathrm{id},0\,\Pi)"] \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "="] & {} \arrow[r, phantom, "="] & {} \\
 & A \arrow[r, "s"] & A
\end{tikzcd} \qquad (+)$$

i. e. satisfying the—PR—equations

$$a + 0 = a : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ as well as}$$
$$a + s\,0 = s(a+n) = (a+n)+1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{s} \mathbb{N}$$

"But" we need more—in particular for **showing** LEIBNIZ type compatibility of iteration Operator $f \mapsto f^{\S}$ with equality, namely *uniqueness* **axiom** for the iterated:

$$(\S!)\quad \frac{\begin{array}{l} f : A \to A \text{ (endo map)} \\ h : A \times \mathbb{N} \to A, \\ h \circ (\mathrm{id}_A, 0) = \mathrm{id}_A \text{ and} \\ h \circ (A \times s) = f \circ h \text{ "as well"} \end{array}}{h = f^{\S} : A \times \mathbb{N} \to A}$$

By adding this uniqueness as **axiom,** iterated map becomes **characterised** by the commutative pentagonal diagram above, and in particular:

Theorem on Compatibility of Iteration with equality between maps: In the presence of all of the **axioms** above, uniqueness **axiom** $(\S!)$ infers

$$(\S_=)\quad \frac{f = g : A \to A}{f^{\S} = g^{\S} : A \times \mathbb{N} \to A}$$

Proof as **Exercise.**

But already for **definition** and characterisation of *multiplication* and moreover for **proof** of "the" laws of Arithmetic, the following *full schema* (pr) of Primitive Recursion is needed, in "pure" categorical form see FREYD 1972, and (then) PFENDER, KRÖPLIN, and PAPE 1994, not to forget its uniqueness clause, (pr!).

This schema reads in *free-variables* notation:

$$\text{(pr)}\quad \frac{\begin{array}{l} g = g(a) : A \to B \text{ PR } \ (\textit{init map}) \\ h = h((a,n),b) : (A \times \mathbb{N}) \times B \to B \ \ (\textit{step map}) \end{array}}{\begin{array}{l} f = f(a,n) = \mathrm{pr}[g,h](a,n) : A \times \mathbb{N} \to B \ \text{ in } \mathbf{PR} \text{ is given such that} \\ f(a,0) = g(a) : A \to B \ \ \text{(init), and} \\ f(a, s\,n) = h((a,n), f(a,n)) : (A \times \mathbb{N}) \to B, \ \ \text{(step)} \\ \qquad \text{as well as} \\ \text{(pr!)} : f \text{ is } \textit{unique} \text{ with these properties.} \end{array}}$$

Basic axiom schema (§) of Primitive Recursion *"constructs"* wanted $f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$, to satisfy equations (pr).

But what about *uniqueness?* Given the *basic iteration equations* (§) above, the following FREYD's schema, here called (FR!), which strengthens *(extends)* (basic) iteration uniqueness, into a schema of *initialised iteration,* entails in fact uniqueness (pr!) of $f = \mathrm{pr}\,[\,g, h\,]$ above, as is shown in FREYD 1972—independently of availability of an internal hom Functor.

This Freyd's Uniqueness schema—completing the **axioms** constituting Theory **PR**—reads

$$
(\mathrm{FR!}) \quad \begin{array}{l}
f : A \to B,\ g : B \to B,\ h : A \times \mathbb{N} \to B, \text{ all in } \mathbf{PR}, \\
h \circ (\mathrm{id}_A, 0 \circ \Pi_A) = f : A \to B, \text{ (init)} \\
h \circ (A \times s) = g \circ h : A \times \mathbb{N} \to B, \text{ (step)} \\
\hline
h = g^{\S} \circ (f \times \mathbb{N}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B,
\end{array}
$$

in form of Freyd's pentagonal diagram:

$$
\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A \times s"] \arrow[d, "f \times \mathbb{N}"'] \arrow[dd, dashed, bend left, "h"] & A \times \mathbb{N} \arrow[d, "f \times \mathbb{N}"'] \arrow[dd, dashed, bend left, "h"] \\
A \arrow[ur, "(\mathrm{id}, 0\,\Pi)"] \arrow[dr, "f"'] \arrow[r, phantom, "="] & B \times \mathbb{N} \arrow[d, "g^{\S}"'] \arrow[r, phantom, "="] & B \times \mathbb{N} \arrow[d, "g^{\S}"'] \\
 & B \arrow[r, "g"] & B
\end{tikzcd}
$$

Freyd's uniqueness diagram (FR!)

Remark: This uniqueness of the *initialised iterated* obviously specialises to **axiom** (§!) of uniqueness of—just "simple"—iterated $f^{\S} : A \times \mathbb{N} \to A$ and so makes that uniqueness axiom redundant.

Problem: Is—conversely—stronger Freyd's uniqueness **axiom** already covered by uniqueness (§!) of "simply" iterated $f^{\S} : A \times \mathbb{N} \to A$? My guess is "no".

An important consequence of full schema (pr) of primitive recursion is the following schema of map definition by case distinction:

$$
\text{(IF)}\quad \frac{\begin{array}{l} \chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \text{ PR predicate,} \\ g, h : A \to B \text{ PR maps} \end{array}}{\begin{array}{l} f = \mathrm{if}[\chi, (g|h)] \ \text{“if } \chi \text{ then } g \text{ else } h\text{”} \\ \ =_{\mathrm{def}}\ \mathrm{pr}[\,h, g \circ \ell\,] \circ (\mathrm{id}_A, \chi) : \\ A \to A \times \mathbb{N} \to B. \end{array}}
$$

Comment: Predicate χ has here still Codomain $\mathbb{N}$, sign : $\mathbb{N} \to \mathbb{N}$. We—Reiter 1982—introduce Object $\mathbb{2} = \{n \in \mathbb{N} \,|\, n < s\,s\,0\}$ only in *Basic* Theory $\mathbf{PRa} = \mathbf{PR} +$ (abstr) of PR, by *predicate-into-Object abstraction.*

Commuting DIAGRAM:

$$
\begin{tikzcd}
 & A \arrow[d, "(\mathrm{id},0)"] \arrow[rr, bend left, "h"] & & \\
A \arrow[r, "(\mathrm{id},\chi)"] \arrow[rrr, bend right, "f"'] & A \times \mathbb{N} \arrow[rr, dashed, "(h|g\circ\ell)"] & & B \\
 & A \times \mathbb{N} \arrow[u, "A\times s"'] \arrow[r, "\ell"'] & A \arrow[ur, "g"'] &
\end{tikzcd}
$$

with $[h, g\,\ell] : A \times \mathbb{N} \to B$ also the induced map out of the sum (coproduct), injections $(\mathrm{id}, 0), A \times s$. Verification immediate.

Free-variable notation:

$$f = f(a) = \text{if}[\chi, (g|h)](a)$$
$$= \begin{cases} g(a) \text{ if } \chi(a) \\ h(a) \text{ if } \neg\,\chi(a) \ \textit{(otherwise).} \end{cases}$$

This terminates presentation (and discussion) of terms and equational **axioms** presenting *fundamental categorical Free Variables Theory* **PR** of *Primitive Recursion.*

1.2 A Monoidal Presentation of Theory PR *

We give here a presentation of Cartesian axioms of fundamental Theory **PR** of Primitive Recursion in terms of *Primitive Recursive diagonal symmetric* half-*Cartesian monoidal structure* ["half" means that the mentioned substitution families, here *terminals* and *projections,* need not be natural transformations] + *Cartesianness* proper, the latter expressed by

Uniqueness of terminal map family $\Pi_A : A \to \mathbb{1}$ and

Godement's <u>equations</u>

$f = \ell_{A,B} \circ (f, g) \equiv \ell_{A,B} \circ (f \times g) \circ \Delta_C :$
$C \xrightarrow{\Delta} C \times C \xrightarrow{f \times g} A \times B \xrightarrow{\ell} A$

as well as

$g = r_{A,B} \circ (f, g) \equiv r_{A,B} \circ (f \times g) \circ \Delta_C :$
$C \xrightarrow{\Delta} C \times C \xrightarrow{f \times g} A \times B \xrightarrow{r} B\,,$

cf. the diagonal symmetric half-terminal Categories of Budach & - Hoehnke 1975, "realised" in particular as (classical) categories of (**sets** and) partial maps.

Main reasons for this alternative presentation are

- Theories $\mathbf{P\widehat{R}a} \sqsubset \mathbf{P\widehat{R}a}\mathbb{X}$ of (genuine) *partial* PR maps to be introduced in chapter 2, inherit the structure of a *PR symmetric diagonal* (just) *half-Cartesian* Theory from *basic* PR Theories $\mathbf{PRa}\mathbb{X} \sqsupset \mathbf{PRa}$ to be discussed below.

  [Theory **PRa** is <u>embedding extension</u> of **PR** by *predicate-into-Object abstraction.*]

- present blow-up of Cartesian structure into a multitude of *simpler* **axioms** makes (later) discussion of *Evaluation and Consistency* easier (?)

Godement's <u>equations</u> "then" are equivalent to *naturality* of projection <u>family</u> for **Bi**Functor $\times : \mathbf{T} \times \mathbf{T} \longrightarrow \mathbf{T}$, $\mathbf{T}$ a Cartesian Theory.

[***In first reading*** *you may* ***skip*** *the remainder of present section as well as subsequent section, on uniqueness of the NNO* $\mathbb{N}$.]

So here is—alternative—presentation of Cartesian part of Theory **PR** as a *PR symmetric diagonal half-terminal Theory with projections:*

Replace in the Cartesian part of presentation of Theory **PR** above *formation of the induced* and its *uniqueness* equation (FM) by introduction of the map constants and schemata producing equations below.

Substitution maps:

$\Pi = \Pi_A : A \to \mathbb{1},$ *terminal map* for Object A,
$\Theta = \Theta_{A,B} : A \times B \xrightarrow{\cong} B \times A,$ *transposition*
$\Delta = \Delta_A : A \to A^2 = A \times A,$ *diagonal, duplicate*
$\ell = \ell_{A,B} : A \times B \to A$ *left projection,*
$r = r_{A,B} \ =_{\mathrm{def}} \ \ell_{B,A} \circ \Theta_{A,B} : A \times B \to B \times A \to B$
right projection.

Fundamental for this structure of our Theory **PR** is generation—enumeration—of *Cartesian product of maps* by **axiom** schema

$$(\times) \quad \frac{f : A \to A',\ g : B \to B' \text{ maps}}{(f \times g) : (A \times B) \to (A' \times B') \text{ map,}}$$

the *Cartesian product* of f and g.

As in case of composition, we need an **axiom** of—Leibniz type—compatibility of Cartesian product of maps with (map) equality, namely

$$(\times_=) \quad \frac{f = f' : A \to A',\ g = g' : B \to B' \text{ maps}}{(f \times g) = (f' \times g') : A \times B \to A' \times B'.}$$

We **break down** wanted GODEMENT equations $\ell \circ (f, g) = f : C \to A \times B \to A$—same for right component—into first:

$$(\Theta - \text{proj}) \quad \frac{A,\ B \text{ Objects}}{r_{A,B} = \Theta_{A,B} \circ \ell_{B,A} : A \times B \to B \times A \to B}$$

and

$$(\Delta - \text{proj}) \quad \frac{C \text{ Object}}{\begin{array}{l} \ell_{C,C} \circ \Delta = \mathrm{id}_C = r_{C,C} \circ \Delta : \\ C \to C \times C \to C, \end{array}}$$

as well as

Naturality of **projections** namely **axiomatic** equalities

$$(\mathrm{nat}_\times) \quad \frac{f : A \to A',\ g : B \to B' \text{ maps}}{\begin{array}{l} \ell_{A',B'} \circ (f \times g) = f \circ \ell_{A,B} : A \times B \to A', \text{ and} \\ r_{A',B'} \circ (f \times g) = g \circ r_{A,B} : A \times B \to B'. \end{array}}$$

As naturality DIAGRAM:

$$
\begin{tikzcd}
A \arrow[r, "f"] & A' \\
A \times B \arrow[u, "\ell"] \arrow[d, "r"] \arrow[r, "f \times g"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & A' \times B' \arrow[u, "\ell"'] \arrow[d, "r"'] \\
B \arrow[r, "g"] & B'
\end{tikzcd}
$$

Cartesian product of maps

We now show—wanted—**availability** of *induced map* $(f, g) : C \to A \times B$ for given $f : C \to A$ and $g : C \to B$ as follows:

Define

$$(f, g) \ =_{\mathrm{def}} \ (f \times g) \circ \Delta_C : C \to C \times C \to A \times B$$

Then this *induced* obviously fullfills (the two) Godement's equations

$$
\begin{tikzcd}
 & A \\
C \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] \arrow[r, "{(f,g)}"] & A \times B \arrow[u, "\ell"'] \arrow[d, "r"] \\
 & B
\end{tikzcd}
$$

Uniqueness of the induced map—Godement—is guaranteed by the earlier Fourman's <u>equational</u> **axiom**—logically simpler than the HORN clause litterally expressing uniqueness.

$$\text{(FM)} \quad \frac{h : C \to A \times B \text{ map}}{\begin{array}{l} h = (\ell_{A,B} \circ h,\ r_{A,B} \circ h) \\ =_{\text{by def}} \ (\ell_{A,B} \circ h \times r_{A,B} \circ h) \circ \Delta_C : \\ C \xrightarrow{\Delta} C \times C \to A \times B \end{array}}$$

These are the **axioms**—and some of the **Theorems** for the **Cartesian** Structure of theory **PR** in construction.

A consequence is **Compatibility** of **induced map** with **equality:** it follows from compatibility of composition and of Cartesian product with equality and from uniqueness of the induced (FOURMAN's equation).

Cartesian Product " $\times$ " introduced above, becomes a **BiFunctor**

$$\times : \mathbf{PR} \times \mathbf{PR} \longrightarrow \mathbf{PR}.$$

This follows from the compatibilities with map equation by uniqueness of the induced map: draw the 4 squares rectangular diagram.

Furthermore follows **Transposition Equation**

$$\Theta_{A,B} = (r_{A,B}, \ell_{A,B}) : A \times B \to B \times A,$$

and **Diagonal Equation**

$$\Delta_A = (\mathrm{id}_A, \mathrm{id}_B) : A \to A \times A,$$

as well as **Naturality** (natural transformation property) of the **substitution** families $\Pi_A, \Theta_{A,B}, \Delta_A$.

These are the (map) term and map-term equality constructors for PR enumeration of the Cartesian part of "fundamental" Theory **PR** of Primitive Recursion, and some of their immediate consequences.

1.3 Uniqueness of the NNO up to Isomorphism*

Comparison with LAWVERE's **sequence defining** description of "an" NNO:

That NNO is given as a diagram $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ —as FREYD's one, but with **defining schema**

$$
(\mathrm{NNO_{FWL}}) \quad \frac{\begin{array}{l} a_0 : \mathbb{1} \to A \text{ a } \textit{point,} \\ f : A \to A \text{ an endo map } \textit{to be iterated} \end{array}}{\begin{array}{l} a = a(n) : \mathbb{N} \to A \textit{ resulting sequence,} \\ a(0) = a_0 : \mathbb{1} \to A, \textit{ start of sequence,} \\ a(n+1) = f(a(n)) : \mathbb{N} \to A \textit{ progress of sequence} \\ + \textbf{ uniqueness} \text{ of such sequence } a : \mathbb{N} \to A, \end{array}}
$$

the latter in DIAGRAM form:

$$
\begin{tikzcd}
 & \mathbb{N} \arrow[r, "s"] \arrow[d, dashed, "a"] \arrow[dr, phantom, "="] & \mathbb{N} \arrow[d, dashed, "a"] \\
\mathbb{1} \arrow[ur, "0"] \arrow[dr, "a_0"'] \arrow[r, phantom, "="] & A \arrow[r, "f"] & A \\
\end{tikzcd}
$$

LAWVERE NNO DIAGRAM

We show that this early NNO schema is obtained from FREYD's schema:

NNO Lemma:

- For $a_0 : \mathbb{1} \to A$ and $f : A \to A$ (antecedent in LAWVERE's NNO schema),

  $$a \;\; =_{\text{def}} \;\; f^{\S} \circ (a_0, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \to A \times \mathbb{N} \xrightarrow{f^{\S}} A,$$

  written with Free Variable $n \in \mathbb{N}$:

  $$a = a(n) = f^n(a_0) : \mathbb{N} \to A$$

  does the job, and **uniquely** so.

- Conversely, LAWVERE's NNO has the properties of an NNO in FREYD's version quoted above—but for his **Proof** of this assertion, FREYD relies on internal hom structure with **axiomatic** exponentiation B^A—coming with **axiomatic** *"internal" (!)* **evaluation** $\epsilon_{A,B} : B^A \times A \to B$—which is available in his context of an (Elementary) Topos.

Proof of the **Lemma:** Consider the following DIAGRAM:

$$\begin{tikzcd}
& \mathbb{1}\times\mathbb{N} \arrow[r, "\mathbb{1}\times s"] \arrow[d, leftrightarrow, "\cong"] \arrow[dr, phantom, "="] & \mathbb{1}\times\mathbb{N} \arrow[d, leftrightarrow, "\cong"] \\
& \mathbb{N} \arrow[r, "s"] \arrow[d, "(a_0{,}\mathrm{id})"] \arrow[dd, dashed, bend right=60, "h"'] \arrow[dr, phantom, "="] & \mathbb{N} \arrow[d, "(a_0{,}\mathrm{id})"'] \arrow[dd, dashed, bend left=60, "h"] \\
\mathbb{1} \arrow[uur, bend left, "(\mathrm{id}{,}0)"] \arrow[ur, "0"] \arrow[r, "(a_0{,}0)"] \arrow[dr, bend right, "a_0"'] & A\times\mathbb{N} \arrow[r, "A\times s"] \arrow[d, "f^\S"] \arrow[dr, phantom, "="] & A\times\mathbb{N} \arrow[d, "f^\S"'] \\
& A \arrow[r, "f"] & A
\end{tikzcd}$$

FREYD to LAWVERE NNO Specialisation DIAGRAM

Here $h = h(n) : \mathbb{N} \to A$ is another *sequence* assumed to fullfill the postdecent above in place of $a : \mathbb{N} \to A$: That

$$a \ =_{\text{by def}} \ f^\S \circ (a_0, \mathrm{id}_\mathbb{N}) \circ \ \cong : \mathbb{1} \times \mathbb{N} \to \mathbb{N} \to A \times \mathbb{N} \to A$$

fullfills postcedent is obvious by above DIAGRAM.

Uniqueness follows as a special instance of Freyd's uniqueness (FR!) : in that schema take as *Initialisation Domain* Object $\mathbb{1}$, as *Iteration Domain* Object A, as *initialisation map* $a_0 : \mathbb{1} \to A$, and as *endo map* to be *iterated* $f : A \to A$.

Commutativity of the DIAGRAM then shows "existence" and uniqueness of LAWVERE's *sequence* $a = a(n) : \mathbb{N} \to A$, from EILENBERG & ELGOT's properties of the iterated $f^\S : A \times \mathbb{N} \to A$—lower two lines of the DI-

AGRAM—combined with Freyd's uniqueness (FR!) : frame DIAGRAM **q.e.d.**

Category theorists like constructions which are *uniquely* given by their defining properties, unique up to *natural isomorphisms,* or—functorial constructions—up to natural equivalence. For the (binary) Cartesian Product with its projection families as *natural map* families this is true by considerations earlier above.

Now what about the natural numbers "Object"

$$\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}\,?$$

This DIAGRAM has the property wanted, property which "should" be called "categoricity": by its LAWVERE *"existence"* and *uniqueness* properties, it is just a ("the") **initial diagram** $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ of form $\mathbb{1} \xrightarrow{a_0} A \xrightarrow{f} A$.

So, **"purely" map theoretically**—not order theoretically—the notion of an NNO *is* "categoric": Within a Cartesian (Map) Theory, NNO $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is unique—up to *natural isomorphism.*

On the map-theoretic level, with "meta" variables a_0, f and h, our "categoricity" of the Natural Numbers Object is ***elementary,*** even of HORN type.

This finishes our axiomatic presentation of theory **PR**, by external HORNER **implications.** Only symmetry and transitivity of map equality, compatibility of composition and of Cartesian product with equality, as well as uniqueness schema (FR!) for the iterated are genuine external HORNER implications: antecedent a conjunction of equations, postcedent an equation. In the other, "equational" cases, the antecedent clauses just state, equationally, fitting of Domains and CoDomains involved in the postcedent equations.

1.4 Introduction of Free Variables

We start with a ("generic") example of **Elimination** of Free Variables by their Interpretation *into (possibly nested)* **projections:**

a distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$ gets the map interpretation

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c) :$$
$$R^3 \ =_{\text{by def}} \ R^2 \times R \ =_{\text{by def}} \ (R \times R) \times R \to R,$$

with *systematic* interpretation of variables:

$$a := \ell\ \ell\,,\ b := r\ \ell\,,\ c := r : R^3 = (R \times R) \times R \to R\,,$$

and infix writing of operations $op : R \times R \to R$ prefix interpreted as

$$\cdot \circ (a\,, + \circ (b, c)) = + \circ (\cdot \circ (a, b)\,, \cdot \circ (a, c)) : R^3 \to R.$$

In form of a commuting diagram:

$$\begin{tikzcd}
 & & (R \times R) \times R \arrow[dll, bend right, "{(a,(b,c))}"', "{(\ell\ell,(r\ell,r))}"] \arrow[ddll, "{(a,b+c)}"] \arrow[ddrr, "{(a\cdot b,a\cdot c)}"'] \arrow[drr, bend left, "{((a,b),(a,c))}", "{((\ell\ell,r\ell),(\ell\ell,r))}"'] & & \\
R \times (R \times R) \arrow[d, "R \times +"'] & & & & (R \times R) \times (R \times R) \arrow[d, "\cdot \times \cdot"] \\
R \times R \arrow[drr, "\cdot"] & & & & R \times R \arrow[dll, "+"'] \\
 & & R & &
\end{tikzcd}$$

An *iterated* $f^{\S} : A \times \mathbb{N}$ may be written in Free-Variables notation as

$$f^{\S} = f^{\S}(a, n) = f^{n}(a) : A \times \mathbb{N} \to A$$

with—canonically—$a := \ell : A \times \mathbb{N} \to A$, and $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Systematic map Interpretation of Free-Variables Equations now seems to be clear by the above examples:

1. extract the common codomain (domain of values), say B, of both sides of the equation (this codomain may be implicit);

2. “expand” operator priority into additional bracket pairs;

3. transform infix into prefix notation, on both sides of the equation;

4. order the (finitely many) variables appearing in the equation, e.g.lexically;

5. if these variables $a_1, a_2, \ldots, a_{\underline{m}}$ range over the Objects $A_1, A_2, \ldots, A_{\underline{m}}$, then fix as common *domain Object* (source of commuting diagram), the Object

$$A = A_1 \times A_2 \times \ldots \times A_{\underline{m}} \ =_{\text{def}} \ (\ldots((A_1 \times A_2) \times \ldots) \times A_{\underline{m}});$$

6. <u>interpret</u> the variables as **identities** resp.(possibly nested) **projections,** will say: **replace,** within the equation, all the occurences of a **variable,** by the corresponding—in general *binary nested*—projection;

7. replace each symbol " 0 " by " $0\ \Pi_D$ " where " D " is the (common) Domain of (both sides) of the equation;

8. insert composition symbol $\circ$ between terms which are not bound together by an *induced map operator* as in (f_1, f_2);

9. By the above, we have the following two-maps-Cartesian-Product rule, forth and back: For

   $a := \ell_{A,B} : (A \times B) \to A$, $b := r_{A,B} : (A \times B) \to B$, and $f : A \to A'$ as well as $g : B \to B'$, the following identity holds:

$$\begin{aligned}
&(f \times g)(a, b) = (f \times g) \circ (\ell_{A,B}, r_{A,B}) \\
&= (f \times g) \circ \mathrm{id}_{(A \times B)} = (f \times g) \\
&= (f \circ \ell_{A,B}, g \circ r_{A,B}) \\
&= (f \circ a, g \circ b) = (f(a), g(b)) : A \times B \to A' \times B';
\end{aligned}$$

10. for free variables $a \in A$, $n \in \mathbb{N}$ interpret the term $f^n(a)$ as the map $f^{\S}(a, n) : A \times \mathbb{N} \to A$.

These 10 interpretation steps transform a (PR) Free-Variables equation into a—variable-free, categorical—equation of theory **PR** :

Elimination of (Free) Variables by Interpretation as *projections,* and vice versa: **Introduction of Free Variables** as *names* for (possibly binary nested) projections. We allow for mixed notation too, all this, for the time being, only in the context of a Cartesian (!) theory **T**.

All of our theories—above and to come—are free from classical, (axiomatic) formal quantification. Free variables equations are understood naively as *universally quantified.* But a Free Variable ($a \in A$) occurring only in the premise of an *implication* takes (in suitable context, see below), the meaning

for any given $a \in A$: premise $(a, \ldots) \implies$ conclusio i. e.

if exists $a \in A$ s. t. premise $(a, \ldots)$, *then* conclusio.

Theory **PR**, formally "enriched" with Free Variables by the above, is—because of Freyd's uniqueness schema (FR!)—a priori stronger than classical (Free-Variables) *Primitive Recursive Arithmetic* **PRA** in the sense of Smorynski 1977. If viewed as a subSystem of **set theory,** of **ZF** say, that **PRA** is stronger than "our" **PR**.

1.5 Goodstein FV Arithmetic

In "Development of Mathematical Logic" (Logos Press 1971) R. L. Goodstein gives four basic uniqueness-rules for Free-Variable Arithmetics. We show here these rules for Theory **PR**, and that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction $a \dot{-} n$.

*For our **evaluation and consistency** considerations below we need from present section equality **predicate** $[a \doteq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{2}$, and that this predicate **defines** map equality, see **equality definability schema** in the center of section. This schema is a consequence of commutativity* $\max(a, b) \ =_{\text{def}} \ a + (b \dot{-} a) = b + (a \dot{-} b) \ =_{\text{by def}} \max(b, a)$ *which is difficult to show and which you may take on faith.*

Basic **GA** operations are *addition* '$+$', *predecessor* 'pre', *truncated subtraction* '$\dot{-}$', [in GOODSTEIN *predecessor* written $\mathrm{pre} := (\_) \dot{-} 1$], as well as *multiplication* '$\cdot$'.

We include[2] into Goodstein's uniqueness rules a "passive parameter" a. These extended rules are derivable by use of Freyd's **Uniqueness Theorem** (pr!), part of *full schema* (pr) of Primitive Recursion which he deduces from his uniqueness (FR!) of the *initialised iterated.*

FREYD 1972 deduces the latter from availability of a Natrual Numbers Object $\mathbb{N}$ in LAWVERE'S sense, and (!) *axiomatic* availability of "higher order" *internal* hom objects with, again axiomatic, *evaluation* map family for these objects, of form $\epsilon_{A,B} : B^A \times A \to B$ within (!) the category considered.

Goodstein's rules with passive parameter:

Let $f, g : A \times N \to N$ be primitive recursive maps, $s : N \to N$

[2] Sandra Andrasek and the author

the successor map $n \mapsto n + 1$ and pre : $N \to N$ the predecessor map, usually written as $n \mapsto n \dot{-} 1$.

Then Goodstein's rules read:

$$\mathrm{U}_1 \quad \frac{f(a, sn) = f(a, n) : A \times \mathbb{N} \to B}{f(a, n) = f(a, 0) : A \times \mathbb{N} \to B}$$

no change by application of successor

infers equality *with value at zero* for f

$$\mathrm{U}_2 \quad \frac{f(a, s\,n) = s\,f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) + n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of successors into $+n$

$$\mathrm{U}_3 \quad \frac{f(a, sn) = \mathrm{pre}\ f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) \dot{-} n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of predecessors into $\dot{-}\, n$

$$\mathrm{U}_4 \quad \frac{\begin{array}{l} f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = g(a,sn) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n) : A \times \mathbb{N} \to \mathbb{N}}$$

uniqueness of map definition by case-distinction

Comment: Theories **PR** and **PRa** allow—within rules U_1 and U_4 above—for replacing $\mathbb{N}$ as a Codomain Object, by an arbitrary object B of **PR** resp.**PRa**.

Rule U_4—of *uniqueness* of maps defined by **case distinction**—is nothing else than *uniqueness* of the *induced map out of the sum* $A \times \mathbb{N} \cong (A \times \mathbb{1}) + (A \times \mathbb{N})$, this sum canonically realised via *injections* $\iota = (\mathrm{id}_A \times 0) : A \times \mathbb{1} \to A \times \mathbb{N}$ as well as—right injection—$\kappa = \mathrm{id}_A \times s : A \times \mathbb{N} \to A \times \mathbb{N}$:

This uniqueness combined with LEIBNIZ' **compatibility** of *induced-map-out-of-a-sum* with map (term) equality, compatibility available in Theories **PR**, **PRa**, and their strengthenings.

Proof of these four rules is straight forward for theories **PR**, **PRa** (and strengthenings), using FREYD's uniqueness (FR!) and uniqueness clause (pr!) of the *full schema of Primitive Recursion* respectively, as follows:

For schema U_1 consider—with free variable $a := \ell : A \times \mathbb{N} \to A$—

$$
\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[dd, "f"', "f(a{,}0)"] & A \times \mathbb{N} \arrow[dd, "f"', "f(a{,}0)"] \\
A \times \mathbb{N} \arrow[ur, "(a{,}0)"] \arrow[dr, "f(a{,}0)"'] & & \\
 & \mathbb{N} \arrow[r, "\mathrm{id}"] & \mathbb{N}
\end{tikzcd}
$$

(FR!)

$$f(a,n) = f = f(a,0).$$

The **postcedent**—U_1—follows by (FR!) for *commutative fill in* of *both* $f : A \times \mathbb{N} \to \mathbb{N}$ and

$$f(a,0) = f \circ (A \times (0 \circ \Pi_{\mathbb{N}})) : A \times \mathbb{N} \to A \times \mathbb{N} \to \mathbb{N}.$$

Proof of U_2 of *"summing up successors":*

$$
\begin{tikzcd}
 & A \times \mathbb{N} \arrow[r, "A\times s"] \arrow[dd, "f"', "f(a{,}0)+n"] \arrow[ddr, bend left=60, "f(a{,}0)+s\,n"] & A \times \mathbb{N} \arrow[dd, "f"', "f(a{,}0)+n"] \\
A \times \mathbb{N} \arrow[ur, "(a{,}0)"] \arrow[dr, "f(a{,}0)"'] & & \\
 & \mathbb{N} \arrow[r, "s"] & \mathbb{N}
\end{tikzcd}
$$

(FR!)

$$f(a,n) = f(a,0) + n$$

The **postcedent**—U_2—again follows from (FR!) by commutativity of DIAGRAM in the present **antecedent.**

[**sorry** for the interruptions in arrow $f(a,0) + sn : A \times \mathbb{N} \longrightarrow \mathbb{N}$ above]

Proof of U_3 is exactly analogous to the above: Replace—in **statement** of U_2 and its **proof**—*stepwise augmentation* $f(a, sn) = s\, f(a,n)$ by *stepwise descent*

$$f(a, s\,n) = f(a,n) \dot{-} 1 \;=_{\text{by def}}\; \mathrm{pre}\; f(a,n) :$$

On right hand side replace *successor* $s : \mathbb{N} \to \mathbb{N}$ by *predecessor* $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ which in turn is PR **defined** by full (!) schema (pr) of Primitive Recursion. In **postcedent**—as well as in **proof**—replace *iterated successor* $a + n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by *iterated predecessor* $a \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

[In GOODSTEIN's *original,* $\mathrm{pre}(n) = n \dot{-} 1 : \mathbb{N} \to \mathbb{N}$ is a **basic,** "undefined" map constant]

We give a **Direct Proof** of U_4.

[A structural **Proof** of U_4 is just by **uniqueness** of *map out of sum* $A \times \mathbb{N} \cong (A \times \mathbb{1}) + (A \times \mathbb{N})$ mentioned above]

We tailor first schema U_4 for convenient use of "full" uniqueness schema (pr!), as follows:

$$U_4 \quad \frac{\begin{array}{l} f = f(a,n),\ f' = f'(a,n) : A \times \mathbb{N} \to B, \\ f(a,0) = f'(a,0) : A \to B, \\ f(a, s\,n) = f'(a, s\,n) : A \times \mathbb{N} \to A \times \mathbb{N} \to B \end{array}}{f = f' : A \times \mathbb{N} \to B.}$$

Now choose—for application of schema (pr!)—*anchor map*

$$g = g(a) \ =_{\text{def}} \ f(a,0) = f'(a,0) :$$
$A \to A \times \mathbb{N} \to B,$ and *step map*
$$h = h((a,n),b) \ =_{\text{def}} \ f(a, s\,n) = f'(a, s\,n) :$$
$$(A \times \mathbb{N}) \times B \xrightarrow{\ell} A \times \mathbb{N} \to B.$$

[Here $h = h((a,n),b)$ does not depend on—formally—"recursive" $b \in B$]

This given, we obtain, via uniqueness clause of *full* schema of PR:

$$(\text{pr!}) \quad \frac{\begin{array}{l} f(a,0) = g(a,0) = f'(a,0), \ \text{(anchor hypothesis)} \\ f(a, s\,n) = h((a,n), f(a,n)) = f'(a, s\,n) \ \text{(step hypothesis)} \end{array}}{f = \text{pr}[\,g, h\,] = f' : A \times N \to B \quad \textbf{q.e.d.}}$$

Together with *reflexivity, symmetry,* and *transitivity* of *equality* $f = g : A \to B$: <u>between</u> maps—as well as with the **defining** PR

equations for fundamental **operations** addition, truncated (!) subtraction, and multiplication—*rules* (axiom schemata) U_1—U_4 above, **define** (categorical) Goodstein's **"Free-Variables Arithmetic"** which we name here **Goodstein Arithmetic, GA**.

We now *quote*—with *passive parameter(s)* made visible—GOODSTEIN's arithmetical equations—together with his **proofs**—first of equations governing truncated subtraction as well as addition: his equations 1. to 2.7, where (recall) $n \dot{-} 1 \;=_{\text{by def}}\; \mathrm{pre} : \mathbb{N} \to \mathbb{N}$ is *predecessor,* defined PR by $\mathrm{pre}\,0 = 0, \quad \mathrm{pre}\; s\,n \;=_{\text{by def}}\; n = \mathrm{id}\; n : \mathbb{N} \to \mathbb{N}$, formally, via full schema (p. r.) of Primitive Recursion as

$$\mathrm{pre} = \mathrm{pr}[(0, 0\,\Pi), r\,l] \circ (\Pi, \mathrm{id}) : \mathbb{N} \to \mathbb{1} \times \mathbb{N} \to \mathbb{N}.$$

The first equation is sort of **basic commutativity** of truncated subtraction, namely

$$(a \dot{-} n) \dot{-} 1 =^{\mathbf{GA}} (a \dot{-} 1) \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \tag{1.}$$
$a \in \mathbb{N}$ free, "passive", $a := \ell : A \times \mathbb{N} \to A$,
$n \in \mathbb{N}$ free, *recursive,* $n := r : A \times \mathbb{N} \to \mathbb{N}$.

Proof:

$$\mathrm{U}_3 \quad \frac{(a \dot{-} s\,n) \dot{-} 1 \;=_{\text{by def}}\; ((a \dot{-} n) \dot{-} 1) \dot{-} 1}{(a \dot{-} n) \dot{-} 1 = ((a \dot{-} 0) \dot{-} 1) \dot{-} n}$$
$$=_{\text{by def}}\; (a \dot{-} 1) \dot{-} n : \mathbb{N}^2 \to \mathbb{N} \quad \textbf{q.e.d.}$$

Next equation is **stepwise simplification rule** for truncated subtraction:

$$s\ a \dot{-} s\ b = a \dot{-} b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad (1.1)$$

here again with free variable $a \in \mathbb{N}$ "passive", and this time free variable $b \in \mathbb{N}$ "active", *recursive,* variable—for **proving** the assertion—again by use of schema U_3 :

$$\mathrm{U}_3 \quad \frac{s\ a \dot{-} s\ s\ b \ =_{\text{by def}}\ (s\ a \dot{-} s\ b) \dot{-} 1}{s\ a \dot{-} s\ b = (s\ a \dot{-} s\ 0) \dot{-} b}$$

$$=_{\text{by def}}\ a \dot{-} b : \mathbb{N}^2 \to \mathbb{N},$$

the latter by **definition** of the *predecessor* "$\dot{-}$ 1" **q.e.d.**

Next simplification, namely **right additive inversion,** seems trivial, it is identic to the corresponding *defining* equation for *integer inversion:*

$$a \dot{-} a = 0 : \mathbb{N} \to \mathbb{N} \qquad (1.2)$$

Proof:

$$\mathrm{U}_1 \quad \frac{\begin{array}{c} s\ a \dot{-} s\ a = a \dot{-} a \\ \text{(by stepwise simplification 1.1 above)} \end{array}}{a \dot{-} a = 0 \dot{-} 0 \ =_{\text{by def}}\ 0 \quad \textbf{q.e.d.}}$$

"Conversely"—subtraction from zero—is **truncation at zero,** namely

$$0 \dot{-} a = 0 : \mathbb{N} \to \mathbb{N}. \qquad (1.3)$$

Proof—not as simple as expected—by schema U_1 :

$$\begin{array}{ll} & 0 \dot{-} s\,a =_{\text{by def}} (0 \dot{-} a) \dot{-} 1 \\ & = (0 \dot{-} 1) \dot{-} a \quad \text{(by 1 above)} \\ & = 0 \dot{-} a : \mathbb{N} \to \mathbb{N} \\ \mathrm{U}_1 & \overline{} \\ & 0 \dot{-} a = 0 \dot{-} 0 = 0 : \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.} \end{array}$$

Subtracting a sum by **sequentially subtracting the summands,** reads:

$$a \dot{-} (b + c) = (a \dot{-} b) \dot{-} c : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \qquad (1.31).$$

Proof, as to be expected by U_3, this time with $(a, b) \in A := \mathbb{N} \times \mathbb{N}$ free, *passive,* and *recursive variable* $c \in \mathbb{N}$ free—the free variables **choosen** as the following (nested) **projections:**

$$a := \ell_{\mathbb{N},\mathbb{N}} \circ \ell_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{\ell} \mathbb{N} \times \mathbb{N} \xrightarrow{\ell} \mathbb{N},$$

$$b := r \circ \ell : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{\ell} \mathbb{N} \times \mathbb{N} \xrightarrow{r} \mathbb{N},$$

put together:

$$(a, b) = \ell_{\mathbb{N}\times\mathbb{N},\mathbb{N}} : A \times \mathbb{N} = \mathbb{N}^2 \times \mathbb{N} \xrightarrow{\ell} A = \mathbb{N}^2,$$

as well as

$$c := r : A \times \mathbb{N} = \mathbb{N}^2 \times \mathbb{N} \xrightarrow{r} \mathbb{N}.$$

This agreed to, we consider the following instance of U_3 :

$$\text{(U}_3\text{)}\quad \frac{\begin{array}{l} a \dot{-} (b + s\,c) \ =_{\text{by def}} \ a \dot{-} s\,(b+c) \quad \text{(definition of } + \text{)}, \\ \qquad =_{\text{by def}} \ (a \dot{-} (b+c)) \dot{-} 1 \quad \text{(definition of } \dot{-} \text{)} \end{array}}{a \dot{-} (b+c) = (a \dot{-} (b+0)) \dot{-} c \ =_{\text{by def}} \ (a \dot{-} b) \dot{-} c.} \quad \textbf{q.e.d.}$$

By use of U_1, **Full Simplification**

$$(a+n) \dot{-} (b+n) = a \dot{-} b : \mathbb{N}^2 \times \mathbb{N} \to \mathbb{N}. \qquad (1.4),$$

now is **proved** as follows, with *passive* $a, b \in \mathbb{N}$, and recursion variable $n \in \mathbb{N}$, all free:

$$\text{(U}_1\text{)}\quad \frac{\begin{array}{l} (a + s\,n) \dot{-} (b + s\,n) \\ \qquad =_{\text{by def}} \ s\,(a+n) \dot{-} s\,(b+n) = (a+n) \dot{-} (b+n), \\ \qquad \text{by } \textit{substitution}\text{—realised essentially as composition} \\ \qquad \text{—of } (a+n) \text{ into } a, \text{ and } (a+n) \text{ into } b \text{ within} \\ \qquad \textit{stepwise simplification equation } 1.1 \text{ above} \end{array}}{(a+n) \dot{-} (b+n) = (a+0) \dot{-} (b+0) \ =_{\text{by def}} \ a+b,}$$

the latter with both a and b—then in turn—recursion variables.

We have right neutrality of $0 : \mathbb{1} \to \mathbb{N}$ with respect to addition. For commutativity $a + b = b + a$ we first need—anchor—**left neutrality**

$$0 + n = n \ [\ =_{\text{by def}} \ n + 0 \ ] \ : \mathbb{N} \to \mathbb{N}, \qquad (2)$$

proved as follows—one of GOODSTEIN's "tricks"—using U_2 :

$$\mathrm{U}_2 \quad \frac{\mathrm{id}_{\mathbb{N}}\, s\, a = s\, a}{\mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a,}$$

and hence

$$a = \mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a = 0 + a : \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.}$$

Commutativity Step for Addition:

$$a + s\, b = s\, a + b : \mathbb{N} \times \mathbb{N} \to B. \qquad (2.1)$$

Proof by U_2 as follows, with free variable $b := r : \mathbb{N}^2 \to \mathbb{N}$ as *recursion variable:*

For $f = f(a, b) \ =_{\mathrm{def}} \ a + s\, b : \mathbb{N} \times \mathbb{N} \to N$:

$$U_2 \quad \frac{f(a, s\, b) \ =_{\text{by def}} \ a + s\, s\, b = s(a + s\, b) = s\, f(a, b) : \mathbb{N}^2 \to \mathbb{N}}{\begin{array}{l} f(a, b) = a + s\, b = f(a, 0) + b \\ =_{\text{by def}} \ (a + s\, 0) + b \ =_{\text{by def}} \ s\, a + b \quad \textbf{q.e.d.} \end{array}}$$

The latter two equations now **give Commutativity of Addition,**

$$a + b = b + a \ [\ =_{\text{by def}} \ +(\Theta_{\mathbb{N},\mathbb{N}}(a, b))\ ] \ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (2.2),$$
$a := \ell : \mathbb{N}^2 \to \mathbb{N}$ passive,
$b := r : \mathbb{N}^2 \to \mathbb{N}$ recursion variable—within **proof.**

Proof via U_4 as follows:

$$\frac{\begin{array}{l} a+0 \ =_{\text{by def}} \ a = 0+a \text{ by (2) above,} \\ a+s\,b = s\,a+b \text{ by (2.1) above (and symmetry of equality)} \end{array}}{\begin{array}{l} a+b \ =_{\text{by def}} \ f(a,b) = g(a,b) \\ =_{\text{by def}} \ s\,a+b : \mathbb{N}^2 \to \mathbb{N} \end{array}} \, U_4$$

q.e.d.

This **gives** also sort of **permutability** for **truncated subtraction**

$$(a \dot{-} b) \dot{-} c = (a \dot{-} c) \dot{-} b : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}.$$

Proof:

$$(a \dot{-} b) \dot{-} c = a \dot{-} (b+c) \text{ by (1.31) above}$$
$$= a \dot{-} (c+b) \text{ by commutativity of addition above}$$
$$= (a \dot{-} c) \dot{-} b \text{ again by (1.31)} \quad \textbf{q.e.d.}$$

From *full simplification* (1.4) and *left neutrality* of zero (2) above with respect to addition we get immediately **"one-term" simplification**

$$(a+n) \dot{-} n = a : \mathbb{N} \times \mathbb{N} \to \mathbb{N}. \qquad (2.3)$$

Associativity of Addition

$$(a+b)+c = a+(b+c) : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N},$$

with free variables

$$a := \ell \circ \ell : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$b := r \circ \ell : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$

both *passive* in **proof,** as well as

$c := r : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ *recursion variable.*

Proof by U_2 : for $f((a,b),c) \ =_{\mathrm{def}} \ a+(b+c) : \mathbb{N}^2 \times \mathbb{N}$:

$$\frac{\begin{aligned} & f((a,b),s\,c) = a+(b+s\,c) = a+s(b+c) \\ & = s(a+(b+c)) = s\,f((a,b),c) \end{aligned}}{\begin{aligned} & a+(b+c) = f((a,b),c) = f((a,b),0)+c \\ & =_{\mathrm{by\,def}} \ (a+(b+0))+c = (a+b)+c : \mathbb{N}^2 \times \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.} \end{aligned}} \ U_2$$

Recall PR **Definition** (and Characterisation) of **Multiplication:**

$$a \cdot 0 = 0 : \mathbb{N} \to \mathbb{N},$$
$$a \cdot (n+1) = (a \cdot n) + a.$$

For this operation, we have not only *annihilation by zero from the right,* but also

Left zero-Annihilation $0 \cdot n = 0 : \mathbb{N} \to \mathbb{N}$.

Proof:

$$U_1 \ \frac{0 \cdot s\,n = (0 \cdot n) + 0 = 0 \cdot n}{0 \cdot n = 0 \cdot 0 = 0 \quad \text{q.e.d.}}$$

For **proving** the other—equational—laws making the Natural Numbers Object $\mathbb{N}$ into a **unitary commutative semiring** with—in addition—truncated subtraction introduced above, with (all) equations combining it with itself and with addition, as well as multiplication distributing over this truncated subtraction, GOODSTEIN's—**derived**—schema V_4 below is helpfull, in an elegant way.

For **proof** of that schema, we rely on

Commutativity of maximum operation:

$$\max(a,b) =_{\text{def}} a+(b \dot{-} a) = b+(a \dot{-} b) =_{\text{by def}} \max(b,a) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$$

Proof[3]: As a first step, we **show**

Diagonal Reduction Lemma for maximum:

$$\max(a,b) = \max(a \dot{-} 1, b \dot{-} 1) + \text{sign}(a+b)$$
$$=_{\text{by def}} \max(a \dot{-} 1, b \dot{-} 1) + (1 \dot{-} (1 \dot{-} (a+b))) :$$
$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Proof: first we show equation

$$\max(a, s\,b) = \max(a \dot{-} 1, s\,b \dot{-} 1) + \text{sign}(a + s\,b) \qquad (1)$$

as follows: from

$$\max(0, s\,b) = s\,b = \max(0,b) + 1 : \mathbb{N} \to \mathbb{N} \qquad (2)$$

and

$$\max(s\,a, s\,b) = s\,\max(a,b) = \max(a,b) + 1$$
$$= \max(s\,a \dot{-} 1, s\,b \dot{-} 1) + \text{sign}(s\,a + s\,b) \qquad (3)$$

we get equation (1) by uniqueness U_4.

Furthermore

$$\max(a,0) = a = (a \dot{-} 1) + \text{sign}(a)$$
$$= \max(a \dot{-} 1, 0 \dot{-} 1) + \text{sign}(a+0) \qquad (4)$$

[3] Goodstein, adapted by G. Myrach

Together with (1) above, this **gives,** again by U_4, the **Lemma.**

From this we get immediately—by substitution—

Opposite Diagonal Reduction Lemma for maximum:

$$\begin{aligned}&\max(b,a) = \max(b \dot{-} 1, a \dot{-} 1) + \mathrm{sign}(b+a)\\ &= \max(b \dot{-} 1, a \dot{-} 1) + \mathrm{sign}(a+b)\end{aligned}$$

We **define** PR—by the full schema (pr) of Primitive Recursion, *availability* part—map

$$\begin{aligned}&\phi = \phi(n,(a,b)) : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N} \ \text{ by}\\ &\phi(0,(a,b)) = 0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \ \text{ and}\\ &\phi(s\,n,(a,b)) = \phi(n,(a,b)) + \mathrm{sign}((a \dot{-} n) + (b \dot{-} n)) :\\ &\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}\end{aligned}$$

We show for this *increment* map ϕ

$$\begin{aligned}&\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b))\\ &= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n,(a,b)) \qquad (5)\\ &\quad \text{as well as}\\ &\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b))\\ &= \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n,(a,b)) \qquad (6)\end{aligned}$$

(same increment).

First we **show** equation (5): Substitution of $(a \dot{-} n)$ for a and $(b \dot{-} n)$ for b within **Reduction Lemma** above gives

$$\begin{aligned}&\max(a \dot{-} n, b \dot{-} n)\\ &= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1) + \mathrm{sign}((a \dot{-} n) + (b \dot{-} n))\end{aligned}$$

Adding $\phi(n,(a,b))$ to both sides of this equation gives

$$\begin{aligned}
&\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a+b)) \\
&= \max((a \dot{-} n) \dot{-} 1, (b \dot{-} n) \dot{-} 1) \\
&\quad + \operatorname{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a+b)) \\
&=_{\text{by def}} \ \max(a \dot{-} s\,n, b \dot{-} s\,n) + \phi(s\,n,(a,b)),
\end{aligned}$$

i. e. equation (5).

We **show** equation (6) as follows: By substitution of $(b \dot{-} n)$ for b and $(a \dot{-} n)$ for a in **Opposite Reduction Lemma** and addition of $\phi(n,(a,b))$ on both sides, we get

$$\begin{aligned}
&\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)) \\
&= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) \\
&\quad + \operatorname{sign}((b \dot{-} n) + (a \dot{-} n)) + \phi(n,(a,b)) \\
&= \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) \\
&\quad + \operatorname{sign}((a \dot{-} n) + (b \dot{-} n)) + \phi(n,(a,b)) \\
&=_{\text{by def}} \ \max((b \dot{-} n) \dot{-} 1, (a \dot{-} n) \dot{-} 1) + \phi(s\,n,(a,b)) \\
&= \max(b \dot{-} s\,n, a \dot{-} s\,n) + \phi(s\,n,(a,b)),
\end{aligned}$$

i. e. equation (6).

From the two **Lemmata,** we get—in both cases—by uniqueness U_1

$$\begin{aligned}
&\max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b)) \\
&= \max(a \dot{-} 0, b \dot{-} 0) + \phi(0,(a,b)) = \max(a,b) + 0 = \max(a,b)
\end{aligned}$$

as well as

$$\begin{aligned}
&\max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)) \\
&= \max(b \dot{-} 0, a \dot{-} 0) + \phi(0,(a,b)) = \max(b,a) + 0 = \max(b,a)
\end{aligned}$$

and hence

$$\max(a,b) = \max(a \dot{-} n, b \dot{-} n) + \phi(n,(a,b)) \text{ as well as}$$
$$\max(b,a) = \max(b \dot{-} n, a \dot{-} n) + \phi(n,(a,b)),$$

and so eventually, by substitution of b into n :

$$\begin{aligned}
&\max(a,b) = \max(a \dot{-} b, b \dot{-} b) + \phi(b,a,b) \\
&= (a \dot{-} b) + \phi(b,(a,b)) \\
&= \max(b \dot{-} b, a \dot{-} b) + \phi(b,(a,b)) \\
&= \max(b,a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}
\end{aligned}$$

i. e. **commutativity of maximum q.e.d.**

This given, we can now **show,** for **GA** (and hence for **PR**), schema

$$\mathrm{V}_4 \quad \frac{\begin{array}{l} f,\ g,\ h : A \times \mathbb{N} \to \mathbb{N}\ \ (\textit{anchor equality}) \\ f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = f(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \\ g(a,sn) = g(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \\ \qquad \text{(same } \textit{progress} \text{ for } f \text{ and } g) \end{array}}{f(a,n) = g(a,n),}$$

variant of U_4.

Rule V_4 can be **derived,** by applying rule U_1 to the distance map

$$\begin{aligned} & d(a,n) = |f(a,n), g(a,n)| = |f(a,n) - g(a,n)| \\ & \quad =_{\text{by def}} \ (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) : \\ & A \times \mathbb{N} \to \mathbb{N}^2 \xrightarrow{+} \mathbb{N} : \end{aligned}$$

$$\begin{aligned} d(a,0) &= (f(a,0) \dot{-} g(a,0)) + (g(a,0) \dot{-} f(a,0)) = 0 \\ d(a,sn) &= (f(a,sn) \dot{-} g(a,sn)) + (g(a,sn) \dot{-} f(a,sn)) \\ &= (f(a,n) + h(a,n)) \dot{-} (g(a,n) + h(a,n)) \\ & \quad + (g(a,n) + h(a,n)) \dot{-} (f(a,n) + h(a,n)) \\ &= (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) \\ & \qquad \text{by } \textbf{absorption} \text{ law for } \dot{-}\ , \ \text{ twice} \\ &= d(a,n) : A \times \mathbb{N} \to \mathbb{N}, \end{aligned}$$

whence, by U_1:

$$\begin{aligned} & d(a,n) = d(a,0) = 0, \ \text{i.e.} \\ & (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) = 0, \ \text{whence} \\ & f(a,n) \dot{-} g(a,n) = 0 = g(a,n) \dot{-} f(a,n) : A \times \mathbb{N} \to \mathbb{N}, \end{aligned}$$

and hence

$$\begin{aligned} f(a,n) &= f(a,n) + (g(a,n) \dot{-} f(a,n)) : A \times \mathbb{N} \to \mathbb{N} \\ &\qquad \text{here enters hypothesis, via } a \dot{-} a = 0 \\ &= \max(f(a,n), g(a,n)) \\ &= \max(g(a,n), f(a,n)) \\ &\qquad \text{by } \textbf{commutative law for maximum,} \\ &\qquad \text{see equation 3 of } \textsc{Goodstein} \\ &\qquad \text{with its fairly elaborated } \textbf{proof} \\ &= g(a,n) + (f(a,n) \dot{-} g(a,n)) \\ &\qquad \text{here enters hypothesis, symmetrically to the above} \\ &= g(a,n) : A \times \mathbb{N} \to \mathbb{N} \quad \textbf{q.e.d.} \end{aligned}$$

Equality Definability Theorem: For any theory **T** *strengthening* (categorical) Free-Variables Theory **PR** of Primitive Recursion, or Theory **PRa** $=_{\text{by def}}$ **PR**+ (abstr), the Theory of PR enriched by *abstractions* $\{A \,|\, \chi\}$ of *predicates* $\chi = \chi(a) : A \to 2 = \{n \in \mathbb{N} \,|\, n < 2\}$—into "additional" *Objects*—we have the following **Equality Definability,** which turns—roughly speaking—internal, "defined" equality of maps into "fundamental" **T**'s equality between maps—the other direction being trivial by reflexivity of $[\, b \doteq_B b' \,] : B^2 \to 2$.

$$\text{(EqDef)}\quad \frac{\begin{array}{l} f = f(a) : A \to B,\ g = g(a) : A \to B \text{ in } \mathbf{T}, \\ \mathbf{T} \vdash\ \mathrm{true}_A = [\, f(a) \doteq_B g(a)\,] : \\ \qquad\qquad A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{\doteq_B} \mathbb{2} \end{array}}{\mathbf{T} \vdash\ f = g : A \to B,\ \text{i.e. } f =^{\mathbf{T}} g : A \to B.}$$

Here, **individual equality** $[\, m \doteq n\,] : \mathbb{N}^2 \to \mathbb{2}$ is **defined** via **weak** (linear) order

$$\begin{aligned} &[\, m \leq n\,] \ =_{\mathrm{def}}\ \neg\,(m \dot{-} n) : \mathbb{N}^2 \to \mathbb{2},\ \text{where} \\ &\neg\, n\ =_{\mathrm{def}}\ 1 \dot{-} n,\ \text{directly PR defined by} \\ &\neg\, 0\ =_{\mathrm{def}}\ 1 \equiv \mathrm{true} : \mathbb{1} \to \mathbb{2} \subset \mathbb{N}, \\ &\neg\, s\, n\ =_{\mathrm{def}}\ 0 \equiv \mathrm{false} : \mathbb{1} \to \mathbb{2} \subset \mathbb{N}. \end{aligned}$$

This weak order on $\mathbb{N}$ is **reflexive** and **transitive.**

Individual ("internal") equality—first on $\mathbb{N}$—then is easily **defined** by

$$[\, m \doteq n\,]\ =_{\mathrm{def}}\ [\, m \leq n\ \wedge\ n \leq m\,] : \mathbb{N}^2 \to \mathbb{2}.$$

Almost by **definition,** the triple $\{\leq,\ \doteq,\ \geq\} : \mathbb{N}^2 \to \mathbb{N}$ fullfills the **law of trichotomy,** and $\max(a, b) : \mathbb{N}^2 \to \mathbb{N}$ above, is characterised as the *maximum* map with respect to the order $[\, a \leq b\,] : \mathbb{N}^2 \to \mathbb{N}$ just introduced—a posteriori.

We now have at our disposition all ingredients for the

Proof of **Equality Definability** schema (EqDef) above:

We begin with the special case $B = \mathbb{N}$: Let $f,\ g : A \to \mathbb{N}$ **T**-maps satisfying the **antecedent** of (EqDef). Then

$$\begin{aligned} f(a) &=^{\mathbf{T}} f(a) + 0 =^{\mathbf{T}} f(a) + (g(a) \dot{-} f(a)) \text{ by antecedent} \\ &=^{\mathbf{T}} \max(f(a), g(a)) \text{ by } \textbf{definition} \text{ of } \max(m,n) \\ &=^{\mathbf{T}} \max(g(a), f(a)) \text{ by } \max \text{ commutativity} \\ &=^{\mathbf{T}} g(a) : A \to B, \text{ symmetric argument, “same way back”.} \end{aligned}$$

The general case—for Codomain Object B—follows, since *individual equality* on (binary) Cartesian Products is—canonically—**defined** *componentwise* using $\wedge$ and restricts to predicative subobjects, both "nicely" **q.e.d.**

[Obviously, Equality Definability holds even for more general theories—with respect to (common) Codomains B, having an *equality predicate* $[\, b \doteq_B b' \,] : B^2 \to \mathbb{N}$ derived as in the above from *fundamental* equality predicate $[\, m \doteq n \,] : \mathbb{N}^2 \to \mathbb{N}$]

This *fundamentals* given, we could continue with properties of the algebraic structure on $\mathbb{N}$, the operations—inclusive order and maximum—already defined.

These properties can be worded as

Algebra, Order and Logic on $\mathbb{N}$:

- $\mathbb{N}$ admits the structure

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "s\,0"'] & \mathbb{N} \arrow[d, ">0"', "\mathrm{sign}"] & \mathbb{N} \times \mathbb{N} \arrow[l, bend right, "\dot{-}"'] \arrow[l, shift right, "+"'] \arrow[l, shift left, "\cdot"] \arrow[dl, shift right, "\leq"'] \arrow[dl, shift left, "\doteq"] \\
& \mathbb{N} &
\end{tikzcd}$$

of a **unary, commutative semiring with zero**—properties of $\dot{-}$, sign : $\mathbb{N} \to \mathbb{N}$ (“positiveness”), order, and equality $\doteq$ see below.

- $\mathbb{N}$ admits a—foundational important—additional algebraic structure, namely **truncated subtraction** $m \dot{-} n : \mathbb{N}^2 \to \mathbb{N}$, with its *simplification properties* when “following” addition—some of these properties see above—and, almost by definition, such that multiplication *distributes* over this kind of subtraction.

  This distributivity will further entail that of multiplication over “full”, not truncated subtraction within

  $$\begin{aligned}
  &\mathbb{Z} \ =_{\mathrm{def}} \ (\mathbb{N} \times \mathbb{N}) / \doteq_{\mathbb{Z}}, \\
  &\qquad \text{with } \textbf{defining} \text{ equality } \textit{predicate} \\
  &[\,(p, q) \doteq_{\mathbb{Z}} (p', q')\,] \ =_{\mathrm{def}} \ [\,p + q' \doteq q + p'\,] : \\
  &\mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{N} \times \mathbb{N} \xrightarrow{\doteq} \mathbb{N}.
  \end{aligned}$$

- $\mathbb{N}$ admits linear **order** $[\,m \leq n\,] : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \subset \mathbb{N}$ as a weak reflexive and transitive *predicate*—this order is PR *decidable.*

- As basic logical structures, $\mathbb{N}$ admits **negation**

$$\neg = \neg\, n : \mathbb{N} \to \mathbb{N},\ \text{as well as}$$

$$\text{sign} = \text{sign}\, n = \neg\,\neg\, n : \mathbb{N} \to \mathbb{N},$$

sign(n) is directly PR **defined** by

$$\text{sign}\, 0 \ =_{\text{def}}\ 0 \equiv \text{false},\ \text{sign}\, s\, n \ =_{\text{def}}\ 1 \equiv s\, 0 :$$

sign $n = [\, n > 0\,] : \mathbb{N} \to \mathbb{N}$ PR decides on *positiveness.*

Furthermore, we have a—fundamental—*equality* ***predicate***

$$[\, m \doteq n\,] \ =_{\text{by def}}\ [\, m \leq n\,] \ \wedge\ [\, m \geq n\,] : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$

which is an *equivalence predicate,* and which makes up a **trichotomy** with strict order

$$[\, m < n\,] \ =_{\text{def}}\ \text{sign}(n \dot{-} m)$$
$$= [\, m \leq n\,] \ \wedge\ \neg\,[\, m \doteq n\,] : \mathbb{N}^2 \to \mathbb{N},$$

Proof of the latter equation as an **Exercise.**

- Object $\mathbb{N}$ admits **definition** of (Boolean) "logical functions" by **truth tables,** as does set $\mathbb{2}$ classically—and below in Theory **PRa** = **PR** + (abstr) of Primitive Recursion with predicate abstraction.

- **Algebra Combined with Order:** As expected, addition is strongly monotonic in both arguments, multiplication is strongly monotonic for both arguments strictly greater zero, and truncated subtraction is weakly monotonic in its first argument and weakly antitonic in its second.

Proof of this **Structure Theorem** is in parts in GOODSTEIN—a first series, concerning $\dot{-}$ and $+$ is quoted above—and for the rest is thought to be a sequence of **Exercises,** solutions to be incorporated possibly into a later version of present ***text.***

Here are some of the **solutions** to these **Exercises** (quoted) already at hand, from S. ANDRASEK:

Theorem: In Free–Variables Arithmetics the **commutative law** for **multiplication:** $n \cdot m = m \cdot n$, holds.

Proof: We prove the commutative law by Peano induction (!). We need the following

Lemma:

(i) $0 \cdot n = 0$

(ii) $sa \cdot n = a \cdot n + n$

Proof:

(i) $0 \cdot 0 = 0$ and
$0 \cdot sn = 0 \cdot (n+1) = 0 \cdot n + 0 = 0 \cdot n = 0 \cdot 0 = 0.$

(ii) We show $f(a,n) := sa \cdot n = g(a,n) := a \cdot n + n$ using V_4: $f(a,0) = g(a,0)$ because for $n = 0$ we get $(sa) \cdot 0 = 0$ as well as $a \cdot 0 + 0 = a \cdot 0 = 0.$

Induction-step:

$$\begin{aligned}
f(a,sn) &= (sa)\cdot(sn) = (a+1)\cdot(n+1)\\
&= (a+1)\cdot n + (a+1) = (sa)\cdot n + sa\\
&= f(a,n) + h(a,n), \qquad \text{with} \qquad h(a,n) := sa\\
g(a,sn) &= a\cdot(sn) + sn = a\cdot(n+1) + (n+1)\\
&= a\cdot n + a + n + 1 = a\cdot n + n + a + 1\\
&= a\cdot n + n + sa\\
&= g(a,n) + h(a,n).
\end{aligned}$$

So V_4 gives $f(a,n) = g(a,n)$ i.e. $sa \cdot n = a \cdot n + n$.

q.e.d.

We continue with the proof of $a \cdot n = n \cdot a$:

From $a \cdot 0 = 0 = 0 \cdot a$ and $a \cdot sn = a \cdot n + n = sn \cdot a$ by the lemma, we conclude $a \cdot n = n \cdot a$ by U_4.

q.e.d.

Theorem In Free–Variable Arithmetics multiplication distributes over addition: $a \cdot (m+n) = a \cdot m + a \cdot n$.

Proof: We prove the law by Peano induction on n (a,m are passive):

Case $n = 0$ is trivial by definition of $+$ and $\cdot$.

From the hypothesis $a\cdot(m+n) = a\cdot m + a\cdot n$ we infer the next step $a\cdot(m+sn) = a\cdot m + a\cdot sn$ by rule V_4 above—with passive parameter

(a, m)—as follows:

$$\begin{aligned} \text{with} \quad & f((a,m),n) := a \cdot (m+n), \\ & g((a,m),n) := a \cdot m + a \cdot n \qquad \text{and} \\ & h((a,m),n) := a \end{aligned}$$

we have

$$\begin{aligned} f((a,m),sn) &= a \cdot (m+sn) = a \cdot (m+(n+1)) \\ &= a \cdot ((m+n)+1) = a \cdot (m+n) + a \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= a \cdot m + a \cdot sn = a \cdot m + a \cdot (n+1) \\ &= a \cdot m + a \cdot n + a \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

So by V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m+n) = a \cdot m + a \cdot n$.

q.e.d.

Theorem: In Free–Variable Arithmetics the associative law holds, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

Proof: We prove the law applying rule V_4 with “active” parameter n and passive parameter (a, m) to

$$\begin{aligned} & f((a,m),n) := a \cdot (m \cdot n), \\ & g((a,m),n) := (a \cdot m) \cdot n \quad \text{and} \\ & h((a,m),n) := a \cdot m. \end{aligned}$$

For $n = 0$ we have: $a \cdot (m \cdot n) = a \cdot 0 = 0$, and on the other hand: $(a \cdot m) \cdot 0 = 0$.

For V_4–step we have:

$$\begin{aligned}
f((a,m),sn) &= a \cdot (m \cdot sn) = a \cdot (m \cdot (n+1)) \\
&= a \cdot (m \cdot n + m) = a \cdot (m \cdot n) + a \cdot m \\
&= f((a,m),n) + h((a,m),n) \\
g((a,m),sn) &= (a \cdot m) \cdot (n+1) = (a \cdot m) \cdot n + a \cdot m \\
&= g((a,m),n) + h((a,m),n).
\end{aligned}$$

By V_4 we get $f((a,m),n) = g((a,m),n)$, i. e. $a \cdot (m \cdot n) = (a \cdot m) \cdot n$.

q.e.d.

Distributivity Theorem: In Free–Variable Arithmetics *multiplication distributes* over *truncated subtraction:* $a \cdot (m \dot{-} n) = a \cdot m \dot{-} a \cdot n$.

Proof by **equality definability,** namely [$f = g$ iff [$f \doteq g$] = *true*], it is sufficient to show

$$f((a,m),n) := a \cdot (m \dot{-} n) \doteq a \cdot m \dot{-} a \cdot n =: g((a,m),n)] = \mathit{true}.$$

We prove this formula by diagonal induction (see Pfender at al. 1994):

Anchoring ($m = 0$ resp. $n = 0$):

$$a \cdot (0 \dot{-} n) = a \cdot 0 = 0 = 0 \dot{-} a \cdot n = a \cdot 0 \dot{-} a \cdot n, \qquad \text{as well as}$$
$$a \cdot (m \dot{-} 0) = a \cdot m = a \cdot m \dot{-} 0 = a \cdot m \dot{-} a \cdot 0.$$

Diagonal–induction step:

$$\begin{aligned}
&f(a,m,n) := a \cdot (m \dot{-} n) \doteq a \cdot m \dot{-} a \cdot n =: g(a,m,n) \\
&\implies f(a,sm,sn) = a \cdot (sm \dot{-} sn) \doteq a \cdot sm \dot{-} a \cdot sn = g(a,sm,sn),
\end{aligned}$$

since

$$\begin{aligned}
f(a, sm, sn) &= a \cdot (sm \mathbin{\dot{-}} sn) = a \cdot (m \mathbin{\dot{-}} n) \\
&= f(a, m, n), \\
g(a, sm, sn) &= a \cdot sm \mathbin{\dot{-}} a \cdot sn = a \cdot (m+1) \mathbin{\dot{-}} a \cdot (n+1) \\
&= (a \cdot m + a) \mathbin{\dot{-}} (a \cdot n + a) \\
&= a \cdot m \mathbin{\dot{-}} a \cdot n \qquad \text{by absorption law for } \dot{-} \\
&= a \cdot (m \mathbin{\dot{-}} n) \\
&= g(a, m, n).
\end{aligned}$$

q.e.d.

Proposition: Addition and multiplication in Free-Variable Arithmetics are weakly monotonous, i. e.

$$\begin{aligned}
m \le n &\implies m \mathbin{\dot{-}} n = 0 \\
&\implies (a+m) \mathbin{\dot{-}} (a+n) \doteq 0 \qquad \text{by absorption law for } \dot{-} \\
&\implies a+m \ \le \ a+n \\
m \le n &\implies m \mathbin{\dot{-}} n = 0 \\
&\implies (a \cdot m) \mathbin{\dot{-}} (a \cdot n) \doteq a \cdot (m \mathbin{\dot{-}} n) \doteq 0 \\
&\implies a \cdot m \ \le \ a \cdot n
\end{aligned}$$

q.e.d.

Integer division with remainder

$$(a \div b, a \text{ rem } b) : \mathbb{N} \times \mathbb{N}_{>0} \to \mathbb{N} \times \mathbb{N}$$

is **defined** uniquely by

$$\begin{aligned} a \div b &= \max\{c \in \mathbb{N} \mid b \cdot c \leq a\} : \mathbb{N} \times \mathbb{N}_{>0} \to \mathbb{N}, \\ a \text{ rem } b &= a \dot{-} (a \div b) \cdot b : \mathbb{N} \times \mathbb{N}_{>0} \to \mathbb{N}. \end{aligned}$$

Here we used *iteration* of binary maximum

$$\max\{a, b\} = \max(a, b) \;=_{\text{by def}}\; a + (b \dot{-} a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

The predicate $a|b : \mathbb{N}_{>0} \times \mathbb{N} \to \mathbb{N}$, a **is a divisor of** b, a **divides** b is **defined** by

$$a|b = [\, b \text{ rem } a \doteq 0\,].$$

Exercise: Construct the Gaussian algorithm for determination of the **gcd** of $a, b \in \mathbb{N}_{>0}$ **defined** as

$$\gcd(a, b) = \max\{c \leq \min(a, b) \mid c|a \wedge c|b\} : \mathbb{N}_{>0} \times \mathbb{N}_{>0} \to \mathbb{N}_{>0}$$

by iteration of mutual rem.

1.6 Peano Induction

Peano's **axioms**—in slightly adapted categorical (Free-Variables) form read—see Reiter 1982 as well as Pfender, Kröplin & Pape—as

Peano Theorem:

- P1: *zero* 0, namely (fundamental) arrow $0 : \mathbb{1} \to \mathbb{N}$,

  is a *natural number.*

- P2: to any natural number n *is associated* a *successor*

  $s\,n \in \mathbb{N}$: this is realised (here) by "fundamental" *successor map*

  $s = s(n) : \mathbb{N} \to \mathbb{N}$.

  Such successor $s(n)$ is *unique:*

  This is given by LEIBNIZ's substitutivity for any map $f = f(a) : A \to B$, namely

$$\frac{f : A \to B \ \ \mathbf{PR}\text{-map}}{\begin{array}{l} a \doteq a' \implies f(a) \doteq f(a') : \\ A \times A \xrightarrow{f \times f} : B \times B \xrightarrow{\doteq_B} \mathbb{N} \end{array}}$$

  here

$$\begin{array}{l} m \doteq n \implies s(m) \doteq s(n) : \\ \mathbb{N} \times \mathbb{N} \xrightarrow{s \times s} : \mathbb{N} \times \mathbb{N} \xrightarrow{\doteq} \mathbb{N}. \end{array}$$

- P3: 0 is not a successor:

  This follows from $sn > 0$, whence $sn \neq 0$, by definition of $m \doteq n$ via $m < n$ via $m \mathbin{\dot{-}} n$.

- P4: equality $s(m) \doteq s(n)$ *implies* $m \doteq n$:

  This is—here—derived *injectivity* (= *monic* property: REITER) of successor map $s : \mathbb{N} \to \mathbb{N}$ which reads in Free Variables[4] as

$$s\,m \equiv s(m) \doteq s(n) \equiv s\,n \implies m \doteq n :$$
$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

- P5: Peano-***Induction***, derived for Theory **PR**—and all of its (conservative) extensions considered here—from *uniqueness* part (pr!) of *full* schema (pr) of Primitive Recursion (Freyd):

$$\text{(P5)}\quad \frac{\begin{array}{l} \varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{N} \quad \text{predicate} \\ \varphi(a, 0) = \mathrm{true}_A(a) \quad (\textit{anchor}) \\ [\,\varphi(a, n) \implies \varphi(a, s\,n)\,] = \mathrm{true}_{A\times\mathbb{N}} \quad (\textit{induction step}) \end{array}}{\varphi(a, n) = \mathrm{true}_{A\times\mathbb{N}} \qquad (\textit{conclusio}).}$$

Proof of Peano Induction Principle (P5) from *full schema* (pr) of Primitive Recursion:[5]

For schema (pr!) choose as anchor map

$$g = g(a) = \varphi(a, 0) = \mathrm{true}(a) : A \to \mathbb{N}, \text{and as step map}$$
$$h = h((a, n), b) = b \vee \varphi(a, s\,n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

By (pr) we get a unique $f = f(a, n) : A \times \mathbb{N} \to \mathbb{N}$ which satifies

$$f(a, 0) = \varphi(a, 0) = \mathrm{true}(a) \quad \text{and}$$
$$f(a, s\,n) = h((a, n), f(a, n)) = f(a, n) \vee \varphi(a, s\,n).$$

[4] FV calculus formalised in section below

[5] Reiter 1982 and Pfender, Kröplin, Pape 1994

This works for $f = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$ as well as for $f = \varphi$, the latter since

$$\begin{aligned}
&\varphi(a, n) \vee \varphi(a, s\,n) \\
&= (\varphi(a, n) \vee \varphi(a, s\,n)) \wedge (\varphi(a, n) \Rightarrow \varphi(a, s\,n)) \\
&\qquad \text{by 2nd hypothesis} \\
&= \varphi(a, s\,n) \quad \text{by boolean tautology} \\
&(\alpha \vee \beta) \wedge (\alpha \Rightarrow \beta) = \beta \quad \textbf{q.e.d.}
\end{aligned}$$

By replacing predicate φ with

$$\psi(a, n) := \bigwedge_{i \le n} \varphi(a, i) : A \times \mathbb{N} \to \mathbb{N}$$

in this **proof** we get

Course of Values Induction Principle:

$$\text{(P5)} \quad \frac{\begin{array}{ll} \varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{N} & \text{predicate} \\ \varphi(a, 0) = \mathrm{true}_A(a) & (\textit{anchor}) \\ [\, \bigwedge_{i \le n} \varphi(a, i) \implies \varphi(a, s\,n) \,] = \mathrm{true}_{A \times \mathbb{N}} & (\textit{induction step}) \end{array}}{\varphi(a, n) = \mathrm{true}_{A \times \mathbb{N}} \quad (\textit{conclusio}).}$$

Here predicate $\bigwedge_{i \le n} \varphi(a, i) : A \times \mathbb{N} \to \mathbb{N}$ is PR **defined** by

$$\begin{aligned}
&\bigwedge_{i \le 0} \varphi(a, i) = \varphi(a, 0) : A \to \mathbb{N}, \\
&\bigwedge_{i \le s\,n} \varphi(a, i) = \bigwedge_{i \le n} \varphi(a, i) \wedge \varphi(a, s\,n) : A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}$$

1.7 Extension by Predicate Abstraction

We extend fundamental theory **PR** *of Primitive Recursion* definitionally *by predicate abstraction Objects* $\{A \,|\, \chi\} = \{a \in A \,|\, \chi(a)\}$. *We get an (embedding) extension* **PRa** $\sqsupset$ **PR** *having all of the expected properties: Just look at* ***Structure Theorem*** *for Theory* **PRa**.

We discuss a PR **abstraction schema** as a basic definitional enrichment of **PR**, into Theory **PRa** of *PR decidable sets and PR maps in between,* decidable subsets of the sets (Objects) of **PR**, i. e.—up to isomorphism—of Objects of form

$$\mathbb{1},\ \mathbb{N}^1 \ =_{\mathrm{def}}\ \mathbb{N},\ \mathbb{N}^{\underline{m}+1} \ =_{\mathrm{def}}\ (\mathbb{N}^{\underline{m}} \times \mathbb{N}).$$

The extension **PRa** is given by adding schemata ($\mathrm{Ext}_{\mathbf{Obj}}$), ($\mathrm{Ext}_{\mathbf{Map}}$), and ($\mathrm{Ext}_{=}$) below to those of (categorical Free-Variables) Theory **PR**. Together they correspond to the *schema of abstraction* in **set theory,** and they are referenced below as *schemata* of *PR abstraction.*

In **chapter** on **Goodstein Arithmetic** we have introduced on NNO $\mathbb{N}$ the following "proto" Boolean structure:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "\mathrm{false}\,\equiv\,0"] \arrow[r, shift right, "\mathrm{true}\,\equiv\,1"'] & \mathbb{N} \arrow[d, ">0"', "\mathrm{sign}"] \arrow[d, bend right=60, "\neg"'] & \mathbb{N}\times\mathbb{N} \arrow[l, shift right, "\vee\,\equiv\,+"'] \arrow[l, shift left, "\wedge\,\equiv\,\cdot"] \arrow[dl, bend left, shift right, "\Rightarrow\,\equiv\,\leq"'] \arrow[dl, bend left, shift left, "\Leftrightarrow\,\equiv\,\doteq"] \\
 & \mathbb{N} &
\end{tikzcd}$$

Definition: A **PR** *predicate,* on an Object A of **PR**, was/is just a **PR** map $\chi : A \to \mathbb{N}$ where, again, *value* 0 of such *predicate* means—logically—false, value(s) $s\,n$ mean—logically—true. Within **PR**, we identify such $\chi : A \to \mathbb{N}$—as a *predicate*—with

$$\mathrm{sign}\,\chi = \mathrm{sign} \circ \chi : A \xrightarrow{\chi} \mathbb{N} \xrightarrow{\mathrm{sign}} \mathbb{N}.$$

Using the Boolean operations on $\mathbb{N}$ above—predicates on $\mathbb{1}$, $\mathbb{N}$, $(N \times \mathbb{N})$—a *Free-Variables Boolean Predicate Calculus* is easily **defined,** making the <u>set</u> of **PR** predicates on (any) Object A of **PR** into a Boolean <u>Algebra.</u>

Our first—predicate-into-Object *abstraction* schema—is now

$$(\mathrm{Ext}_{\mathbf{Obj}}) \quad \frac{\chi : A \to \mathbb{N} \text{ a } \mathbf{PR}\text{-predicate}}{\{A \,|\, \chi\} \text{ Object (of emerging Theory } \mathbf{PRa})}$$

Set $\{A \,|\, \chi\} \subseteq A \cong \mathbb{N}^{\underline{n}}$ may be written alternatively, with *bound* variable a, as

$$\{A \,|\, \chi\} = \{a \in A \,|\, \chi(a)\}.$$

$\{A \,|\, \chi\}$ is just another name for the (external <u>code</u>) $\chi \in \mathbf{PR} \subset \underline{\mathbb{N}}$, $\underline{\mathbb{N}}$ designating the set of external, “naive” natural numbers.

The *maps* of $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$—between given **PRa** sets—arise by schema

$$(\mathrm{Ext}_{\mathbf{Map}}) \quad \frac{\begin{array}{l} \{A \,|\, \chi\},\ \{B \,|\, \psi\} \ \mathbf{PRa}\text{-Objects,} \\ f : A \to B \text{ a } \mathbf{PR}\text{-map,} \\ \mathbf{PR} \vdash\ \chi(a) \implies \psi\, f\,(a), \text{ i.e.} \\ [\,\chi \implies \psi \circ f\,] =^{\mathbf{PR}} \mathrm{true}_{\mathbb{N}} : \mathbb{N} \xrightarrow{!} \mathbb{1} \xrightarrow{1} \mathbb{N} \end{array}}{f \text{ is a } \mathbf{PRa}\text{-map } f : \{A \,|\, \chi\} \to \{B \,|\, \psi\}}$$

In particular, if for predicates $\chi',\ \chi'' : A \to \mathbb{N}$

$$\mathbf{PR} \vdash\ \chi'(a) \implies \chi''(a) : A \to \mathbb{N} \times \mathbb{N} \to 2\!\!2 \ =_{\text{def}}\ \{n \in \mathbb{N} \,|\, n \le 1\}$$

then $\mathrm{id}_A : \{A \,|\, \chi'\} \to \{A \,|\, \chi''\}$ in **PRa** is called an *inclusion,* and written $\ \subseteq : A' = \{A \,|\, \chi'\} \to A'' = \{A \,|\, \chi''\}$ or $A' \subseteq A''$.

Dangerous Bound: For predicate (terms!) $\chi,\ \psi : A \to \mathbb{N}$ such that $\mathbf{PR} \vdash\ \chi = \psi : A \to \mathbb{N}$—logically: such that $\mathbf{PR} \vdash\ [\chi \iff \psi]$ we have

$$\{A \,|\, \chi\} \subseteq \{A \,|\, \psi\}, \text{ and } \{A \,|\, \psi\} \subseteq \{A \,|\, \chi\},$$

but—in general—not *equality of Objects.* We only get in this case

$$\mathrm{id}_A : \{A \,|\, \chi\} \xrightarrow{\cong} \{A \,|\, \psi\}$$

as an **PRa** *isomorphism.*

This in contrast to earlier **definitions** of Theory **PRa** : We maintain here—by general reasons in categorical framework—distinction of *Objects* which differ in their formal "presentation" as terms of the Object Language of a—categorical—Theory (!).

[This is to extend formal distinction—within Theory **PR**—of Objects $((A \times B) \times C) \cong (A \times (B \times C))$ as well as of $(A \times B) \cong (B \times A)$ which are "only" (naturally) isomorphic, by transformations $(\mathrm{ass}) = (\ell\,\ell, (r\,\ell, r))$ and $\Theta = (r, \ell)$ respectively]

So *inclusion* $\mathrm{id}_A : \{A \,|\, \chi'\} \subset \{A \,|\, \chi''\}$ above is—formally—only an inclusion up to isomorphism.

A posteriori, we introduce now formally *truth Algebra* $2\!\!2$ as

$$2\!\!2 \ =_{\text{def}}\ \{n \in \mathbb{N} \,|\, n \le 1\ : \mathbb{N} \to \mathbb{N}\},$$

with Boolean operations above restricting—in CoDomain and Domain—to $\mathbb{2}$ resp.$\mathbb{2} \times \mathbb{2} \ =_{\text{by def}} \ \{(m,n) \in \mathbb{N} \times \mathbb{N} \,|\, m,n \leq s\,0\}$, by definition of Cartesian Product of Objects within **PRa**. **PRa**-maps (with common **PRa** Domain and CoDomain) are considered to be equal if their values are equal on their defining *Domain predicate.* This is expressed by (defining) schema

$$(\mathrm{Ext}_=) \quad \frac{\begin{array}{l} f,\, g : \{A \,|\, \chi\} \to \{B \,|\, \psi\} \ \textbf{PRa}\text{-maps}, \\ \textbf{PR} \vdash \ \chi(a) \implies f(a) \doteq_B g(a) \end{array}}{f = g : \{A \,|\, \chi\} \to \{B \,|\, \psi\},}$$

explicitely:

$$\begin{array}{l} f =^{\textbf{PRa}} g : \{A \,|\, \chi\} \to \{B \,|\, \psi\}, \ \text{also noted} \\ \textbf{PRa} \vdash \ f = g : \{A \,|\, \chi\} \to \{B \,|\, \psi\}. \end{array}$$

Structure Theorem for Theory **PRa** of *Primitive Recursion with Predicate Abstraction:*[6]

(i) **PRa** inherits (associative) **map composition** and identities from **PR**.

(ii) **PRa** has **PR** fully **embedded**—as a category—by

$$\langle f : A \to B \rangle \mapsto \langle f : \{A \,|\, \mathrm{true}_A\} \to \{B \,|\, \mathrm{true}_B\} \rangle$$

[6] cf. Reiter 1980

(iii) **PRa** has **Cartesian Product**

$$\{A \,|\, \chi\} \times \{B \,|\, \psi\} \;=_{\text{def}}\; \{A \times B \,|\, \chi \wedge \psi : A \times B \to \mathbb{N} \times N \to \mathbb{N}\},$$

with *projections*—and universal property—inherited from **PR**.

(iv) Object $\mathbb{2}$ comes as *sum* $\mathbb{1} \xrightarrow{\text{false}} \mathbb{2} \cong \mathbb{1} + \mathbb{1} \xleftarrow{\text{true}} \mathbb{1}$ over which Cartesian product $A \times \_$ *distributes.*

This allows in fact for the usual **truth-table definitions** of (all) *boolean operations* on Object $\mathbb{2}$ and—distributivity—for PR (!) map **definition** by **case distinction.**

(v) The embedding $\sqsubset$: **PR** $\longrightarrow$ **PRa** is a **Cartesian Functor**—it preserves Products, and their *Cartesian* universal property, with respect to the *projections* inherited from **PR**.

(vi) **PRa** has **Extensions** of its *predicates,* namely

$$\begin{aligned}&\mathrm{Ext}\,[\,\psi : \{A \,|\, \chi\} \to \mathbb{2}\,] \;=_{\text{def}}\; \{A \,|\, \chi \wedge \psi\} \subseteq \{A \,|\, \chi\},\\ &\qquad \text{characterised as } (\mathbf{PRa})\text{-}\textit{equalisers}\\ &\mathrm{Equ}\,(\chi \wedge \psi,\ \mathrm{true}_A) : \{A \,|\, \chi\} \to \mathbb{2}\end{aligned}$$

[mutatis mutandis: within Theory **PRa**, we confound deliberately predicates $\chi = \mathrm{sign} \circ \chi : A \to \mathbb{N} \to \mathbb{N}$ with these predicates seen as maps $\chi : A \to \mathbb{2}$. We may do this because of definition of equality in **PRa**, of $\mathrm{sign}(n) \in \mathbb{2} \subset \{\mathbb{N}|\mathrm{true}_N\}$, and of the notion of a predicate in **PR** and **PRa**. The embeddding further confounds Object A of **PR** with $\{A|\mathrm{true}_A\}$, in particular $\mathbb{N}$ with $\{\mathbb{N}|\mathrm{true}_\mathbb{N}\}$. Domain of $\chi \wedge \psi = \wedge \circ (\chi, \psi) : A \to \mathbb{2} \times \mathbb{2} \to \mathbb{2}$ is A]

(vii) **PRa** has *all* **equalisers,** namely equalisers

$$\mathrm{Equ}[\,f,g\,] \;=_{\mathrm{def}}\; \mathrm{Ext}[\,\dot{=}_B \circ (f,g) : A' \to B' \times B' \xrightarrow{\dot{=}} \mathbb{2}\,]$$

of arbitrary **PRa** map pairs $f,g : A' = \{A \,|\, \chi\} \to B' = \{B \,|\, \psi\}$, and hence all finite (projective) **limits,** in particular **pullbacks,** which we will rely on later.

[I prefer this "set theoretical" way to construct extension sets out of the Cartesian Category structure of fundamental theory **PR**, and then I construct equalisers and the other finite limits on this basis. Another possibility—ROMAN(?)—is to add equalisers as *undefined notion* and to construct directly from these and Cartesian product. The relation between (vi) and (vii) is best understood set theoretically: use free variable argument chase, and recall set theoretical definition of an equaliser]

The embedding **preserves** such limits as far as available already in **PR**. Equality *predicate* extends to Cartesian Products componentwise as

$$[\,(a,b) \doteq_{A\times B} (a',b')\,] \;=_{\mathrm{def}}\; [\,a \doteq_A a'\,] \wedge [\,b \doteq_B b'n\,] : (A \times B)^2 \to \mathbb{2},$$

and to (predicative) subObjects $\{A \,|\, \chi\}$ by restriction.

(viii) Arithmetical structure extends from **PR** to **PRa**, i. e. **PRa** admits the *iteration* schema as well as FREYD's *uniqueness* schema: the iterated

$$f^{\S} : \{A \,|\, \chi\} \times \{\mathbb{N} \,|\, \mathrm{true}_{\mathbb{N}}\} \to \{A \,|\, \chi\}$$

is just the *restricted* **PR**-map $f^{\S} : A \times \mathbb{N} \to A$, the uniqueness schema(ta) follow from **definition** of $=^{\mathbf{PRa}}$ via **PRa**'s schema $(\mathrm{Ext}_=)$ above.

(ix) In particular, our *equality predicate* $\dot{=}_A : A^2 \to \mathbb{N}$, restricted (above) to subObjects $A' = \{A \,|\, \chi\} \subseteq A$, inherits all of the (characteristic) general properties of equality on $\mathbb{N}$ and the other *fundamental Objects.*

(x) **Countability:** Each **fundamental** Object A i. e. A a (bracketed) power of $\mathbb{N} \equiv \{\mathbb{N} \,|\, \mathrm{true}_{\mathbb{N}}\}$, admits, by Cantor's isomorphism

$$\mathrm{cantor} = \mathrm{cantor}_{\mathbb{N}\times\mathbb{N}}(n) : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N},$$

a—retractive—count $\mathrm{cantor}_A(n) : \mathbb{N} \twoheadrightarrow A$.

Problem: For which predicates $\chi : A \to \mathbb{2}$ (A fundamental) does Theory **PRa** admit a (then PR, retractive) *count*

$$\mathrm{count} = \mathrm{count}_{\{A \,|\, \chi\}}(n) : \mathbb{N} \twoheadrightarrow \{A \,|\, \chi\}?$$

The difficulty is seen already in case $\emptyset_A \ =_{\mathrm{by\,def}}\ \{A \,|\, \mathrm{false}_A\}$. A sufficient condition is $\{A \,|\, \chi\}$ to come with a *point,* $a_0 : \mathbb{1} \to \{A \,|\, \chi\}$. "But" their may be non-empty Objects without points in "suitable" theories.

Proof of Theorem: The extensions $\{A \,|\, \chi\}$ are just defined *formally,* by

$$\{A \,|\, \chi\} = \{A \,|\, \chi : A \to \mathbb{N}\} \ =_{\mathrm{def}}\ \langle\, \chi : A \to \mathbb{N} \,\rangle \text{ in } \mathrm{Obj}(\mathbf{PRa}).$$

A preliminary version of **PRa** is constructed by extending canonically **PR**'s structure to the new Objects and maps as introduced by the first two schemata. This structure is then factorised by—formally—identifying maps with (formally!) common Domain and CoDomain, which are **PR**-equal on $\mathrm{sign} \circ \chi : A \to \mathbb{N}$—according to **PRa**-equality schema ($\mathrm{Ext}_=$).

The **proof** that this "equality" $=^{\mathbf{PRa}}$ is in fact a *congruence* with respect to the structure of an arithmetical theory is long, but again straight forward, the details were carried out by Reiter 1980, as well as those proving that extension **PRa** of **PR** has all finite limits, in particular extensions of predicates, equalizers, and pullbacks. And—further—of the characteristic properties of equality predicate restriction to predicative subsets $A' = \{A \mid \chi\} \subseteq A \equiv \{A \mid \mathrm{true}_A\}$.

The last statement of the **Theorem** follows immediately from Cantor's (Free Variable) PR isomorphism

$$\mathrm{cantor} = \mathrm{cantor}_{\mathbb{N}\times\mathbb{N}} : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N}.$$

Remark: In parallel to the above, Reiter 1980 shows that arithmetical theories allow for a formal extension by *quotient* sets A/ρ, $\rho : A \times A \to 2$ an *equivalence predicate* on A, the resulting theory being arithmetical again, and having extensions of predicates if this is the case for the original theory.

In fact, already $\mathbf{PRa} = \mathbf{PR} +$ (abstr) has these quotients, in form $A/\rho \ =_{\mathrm{def}} \ \{a \in A \mid a \doteq_A \bar{a}\}$ where $\bar{a} \ =_{\mathrm{def}} \ \min\{\tilde{a} \leq_A a \mid \tilde{a}\, \rho\, a\}$ is the minimal representant of the equivalence class of a, minimal with respect to the linear well order $\leq_A : A \times A \to 2$ on A which is given by Cantor's isomorphism $\mathrm{cantor}_A : \mathbb{N} \xrightarrow{\cong} A$, A a (nested) binary power of $\mathbb{N}$, and its CoDomain restriction to subObjects $A' = \{A \mid \chi\}$ in **PRa**. In formal terms:

PRa admits the following schema of forming **Quotients** by **equivalence predicates:**

$$(\mathrm{QuotPred})\quad \frac{\rho : \{A \mid \chi\}^2 \to \mathbb{2} \text{ an equivalence predicate in } \mathbf{PRa}}{\begin{array}{l}\{A \mid \chi\}/\rho \ =_{\mathrm{def}}\ \{a \in A \mid a \doteq_A \min\{\tilde{a} \leq_A a \mid \tilde{a}\,\rho\,a\}\}, \\ \text{together with } \textit{quotient map} \\ \mathrm{nat}_\rho = \mathrm{nat}_\rho(a) \ =_{\mathrm{def}}\ [a]_\rho : \{A \mid \chi\} \twoheadrightarrow \{A \mid \chi\}/\rho \end{array}}$$

has the universal properties of a **coequaliser** of **PRa** pair $\{(a', a'') \in \{A \mid \chi\}^2 \mid a'\,\rho\,a''\} \xrightarrow{\subseteq} A \times A \underset{r}{\overset{\ell}{\rightrightarrows}} A.$

Here $[a]_\rho \ =_{\mathrm{def}}\ \min\{a' \leq_A a \mid a'\,\rho\,a\} : \{A \mid \chi\} \twoheadrightarrow \{A \mid \chi\}/\rho$ is **defined** as **minimal representant.** Latter pair is—becomes—the (canonical) **Kernel Pair** $\mathrm{KP}[\mathrm{nat}_\rho]$ of **quotient** $\mathrm{nat}_\rho : \{A \mid \chi\} \twoheadrightarrow \{A \mid \chi\}/\rho$.

Dangerous Bound: *Generation* of an **equivalence** out of an (arbitrary) $\{A \mid \chi\}$ *predicate* $\sigma : A^2 \to \mathbb{2}$ gives in general only an equivalence *relation* $\overline{\sigma} : D_{\overline{\sigma}} \to A^2$, *not* a generated equivalence *predicate* $\overline{\sigma} : A^2 \to \mathbb{2}$: In general we have—a priori—no *decision,* if given “element pairs” in A^2 admit a joining σ-*transitivity chain.*

These chains are PR *enumerable,* and this enumeration “just” gives an *enumeration* of the **relational** *transitive hull* of a given *predicate*—and also that of a given *relation*—within Domain A^2, A Object of Theory **PRa** or of a strengthening **S** of **PRa**.

The **Problem** of integrating—*constructively*—**Quotients** by Equivalence **Relations** into a PR theory, is somewhat involved:

Reiter has formally added such quotients to Cartesian PR Theories with predicate abstraction (and before quotients by equivalence *predicates*), and obtains a Cartesian PR Theory with the original one embedded—preserving its structure—and gets this way a theory **SQ** which has in addition **Quotients** by those **equivalence Relations** which are "brought in" by the original Theory.

Iterating—externally (!)—this *stepwise* "closure" by Quotients of equivalence relations, one arrives at a certain **Closure** of—e.g.—Theory **PRa** under some important *structural requirements:*

This Closure $\mathbf{S\bar{Q}}$ is a Cartesian PR Theory, it has (Universal) **Sums** ("Coproducts"), and **Quotients** by **Equivalence Relations,** as well as—**Conjecture** at the moment—the usual, enumerative ***Construction*** of Quotients by arbitrary Relations.

But on the *projective-Limit* side, we will get—globally in the **Theory Hull,** "just" the *Cartesian Structure*—$\mathbb{1}$ as *terminal object* and *Cartesian Product.* As far as I can see now, we will have **Equality predicates**—and then **Equalisers**—a priori only in *basic* Theories **PRa** and strengthenings **S**, but nevertheless with their *universal properties* preserved by the embedding into the **Hull.**

For a long while, we will need just **Quotients** by those Equivalence **Relations** which "come in" from Theory **PRa** resp.its **strengthenings,** say **S** in general. We call this—Reiter's—Theory **PRaQ** = **PRa** + Quot, **SQ** = **S** + Quot in general.

For the remainder of this book, let **S** be Theory **PRa** or a *strengthening* of **PRa** by additional axioms e.g.later: suitable axioms stating *impossibility of infinite descent* in *Ordinals O* like $\mathbb{N}^2$ or $\mathbb{N}[\omega] \subset \mathbb{N}^*$.

What we need here is that such a theory **S** has extensions of all of its predicates, as well as limits of finite **S**-Diagrams, in particular **S** pullbacks. And that it admits—as a **Cartesian theory**—canonical Interpretation of Free Variables as identities resp.(possibly nested) projections.

As a fundamental Theorem for such Extensions **S** of **PR**, we get the following schema:

Equality Definability Theorem: An arithmetical Theory **S** of form above admits the following schema (EquDef):

$$\text{(EquDef)} \quad \frac{\begin{array}{l} f, g : A \to \mathbb{N} \text{ in } \mathbf{S}, \\ \mathbf{S} \vdash \ f(a) \doteq f(b) : \mathbb{N}^2 \to \mathbb{2} \end{array}}{\begin{array}{l} \mathbf{S} \vdash \ f = g : A \to \mathbb{N}, \ \textit{algebraically:} \\ f =^{\mathbf{S}} g : A \to \mathbb{N}. \end{array}}$$

Equality Definability extends to **S**-map pairs $f, g : A \to B$ with (common) Codomain a (finite) cartesian product B of Objects $\mathbb{N}$, a *fundamental* Object—in **PR**—or “even” B an Object of Theory **PRa**.

Proof: Equality *predicate* $\doteq$ on $\mathbb{N}$ has been **defined** above as

$$[\, m \doteq n \,] \ =_{\text{def}} \ \neg\, |m - n| \ =_{\text{by def}} \ \neg[(m \dot{-} n) + (n \dot{-} m)] : \mathbb{N}^2 \to \mathbb{2},$$

using *truncated subtraction* $(m \dot{-} n) : \mathbb{N}^2 \to \mathbb{N}$, and negation $\neg(a) : \mathbb{N} \to \mathbb{2}$. *Substitution* (realised as composition), of $f : A \to \mathbb{N}$ into m, and $g : A \to \mathbb{N}$ into n gives:

$$[\, f \doteq g \,] \ =_{\text{by def}} \doteq \circ\, (f, g) =^{\mathbf{S}} \neg \,\circ\, |f - g| : A \to \mathbb{N}^2 \to \mathbb{N} \to \mathbb{N},$$

and further, with commutativity of the max function[7] namely

$$\max(m,n) \ =_{\text{def}} \ m + (n \dot{-} m) = n + (m \dot{-} n) \ =_{\text{by def}} \ \max(b,a),$$

by LEIBNIZ substitution property for theory **S**—below—for equality predicate $[\,a \doteq b\,] : \mathbb{N}^2 \to \mathbb{2}$:

$$\frac{\begin{array}{l} f, g : A \to \mathbb{N} \text{ in } \mathbf{S}, \\ \mathbf{S} \vdash \ [\,f \doteq g\,] : \mathbb{N}^2 \to \mathbb{2} \end{array}}{\begin{array}{l} |f - g| = 0 = |g - f|, \text{ hence} \\ (f \dot{-} g) = 0 = (g \dot{-} f), \text{ and hence} \\ f = f + (g \dot{-} f) \ =_{\text{by def}} \ \max(f,g) \\ = \max(g,f) \ =_{\text{by def}} \ g + (f \dot{-} g) \\ = g : A \to \mathbb{N}. \end{array}}$$

The case of B an arbitrary **PRa**-Object follows from the above by **definition** of equality predicate $[\,b \doteq_B b'\,] : B^2 \to \mathbb{2}$ via *conjunction* of equality predicates on the *components* and *restriction* to predicate extensions $\{C \mid \psi\}$ **q.e.d.**

Leibniz Substitutivity for this (family of) **equality** *predicates* $\doteq_A : A^2 \to \mathbb{2}$ reads:

$$(Sub_{\doteq}) \quad \frac{f : A \to B \text{ in } \mathbf{S} \text{ i.e. in } \mathbf{PRa}}{\mathbf{S} \vdash \ [\,a \doteq a'\,] \implies [\,f(a) \doteq_B f(a')\,]}$$

[7] see chapter on Goodstein Arithmetic above

Proof by external structural induction on $f : A \to B$, with a **PRa** internal Peano Induction for the cases of f an iterated $f = g^{\S} : B \times \mathbb{N} \to B$, details see Pfender & Kröplin & Pape 1994.

There is a "bottom up" characterisation of the "top down" recursively introduced **iteration** $f^{\S} = f^{\S}(a, n)$ of an endo map $f : A \to A$, in an arbitrary PR theory **T**—**T** not necessarily having *predicative* equality on A—, usefull in particular for the constructive evaluation business to come in later work:

Bottom up Resolution Lemma for Primitive Recursion: For any PR theory **T**, and a **T** endo map $f : A \to A$ we have iterated $f^{\S} : A \times \mathbb{N} \to A$ characterised by

$$f^{\S}(a, 0) = a = \mathrm{id}_A : A \to A \quad \text{(anchor)},$$

as before, **and**

$$f^{\S}(a, s\,n) = f^{\S}(f(a), n). \quad (\text{step})_{\text{bottomup}}$$

This alternative characterisation *resolves* iteration $f^{\S}$ into a bottom up, iterative *substitution* of argument so far calculated into the endo to be iterated, beginning with given initial argument.

[Algebraically, equality of the two "sides of the medal" is nothing else then *iterated associativity* of <u>composition</u> of theory **S**]

Proof of **Lemma** by *uniqueness* clause (pr!) of Freyd's (categorical version of) *full schema of Primitive Recursion:*

Abbreviate left and right hand side of the assertion by

$$\begin{aligned}
&\mathrm{lhs} = \mathrm{lhs}(a,n) := f^{\S}(a, s\,n) \ =_{\text{by def}} \ f \circ f^{\S}(a,n) : A \times \mathbb{N} \to \mathbb{N} \\
&\qquad \text{as well as} \\
&\mathrm{rhs} = \mathrm{rhs}(a,n) := f^{\S}(f(a),n) : A \times \mathbb{N} \to A.
\end{aligned}$$

Define the (common) *anchor* as

$$\mathrm{anc} = \mathrm{anc}\,(a) \ =_{\text{def}} \ f(a) = \mathrm{lhs}(a,0) = \mathrm{rhs}(a,0) : A \to A,$$

and *step map* for schema (pr) as

$$\mathrm{step} = \mathrm{step}((a,n),a) \ =_{\text{def}} \ f(a) : (A \times \mathbb{N}) \times A \xrightarrow{r} A \xrightarrow{f} A.$$

This *step* works as step-map for definition—by schema (pr)—of both lhs and rhs :

$$\begin{aligned}
&\mathrm{lhs}(a, s\,n) = f \circ f^{\S}(a, s\,n) \\
&\quad =_{\text{by def}} \ f \circ f \circ f^{\S}(a,n) = \mathrm{step}((a,n), \mathrm{lhs}(a,n)) : A \times \mathbb{N} \to A, \\
&\mathrm{rhs}(a, s\,n) = f^{\S}(f(a), s\,n) \\
&\quad =_{\text{by def}} \ f \circ f^{\S}(f(a),n) = \mathrm{step}((a,n), \mathrm{rhs}(a,n)) : A \times \mathbb{N} \to A.
\end{aligned}$$

"Full" Uniqueness Schema (pr!) of Primitive Recursion then gives us, from these *anchor* and *step* equations, (derived) equality

$$\mathbf{S} \vdash \ f^{\S}(a, s\,n) = \mathrm{lhs}(a,n) = \mathrm{rhs}(a,n) = f^{\S}(f(a),n) : A \times \mathbb{N} \to \mathbb{2} \quad \textbf{q.e.d.}$$

Remarks:

(i) A **PRa**-map $f : \{A \,|\, \chi\} \to \{B \,|\, \psi\}$ can be viewed as a *defined partial* **PR** map from A to B with values in ψ : Set of *defined arguments,* namely $\{a \in A \,|\, \chi(a)\}$ is PR *decidable.* By **definition** of

PRa's equality, **PR**-map $f : A \to B$ "doesn't care" about arguments a in the *complement* $\{a \in A \,|\, \neg\, \chi(a)\}$.

So wouldn't it be easier to realise this view to *defined partial maps* just by throwing the *undefined arguments* into a *waste basket* $\{\perp\}$ say?

But where to place this waste basket, this for each Codomain Object B? The fundamental Objects have a zero-vector as a candidate. For example we could interprete truncated subtraction as a *defined partial* map

$$a \dot{-} b : \{(m,n) \in \mathbb{N} \times \mathbb{N} \,|\, m \geq n\} \to \mathbb{N},$$

and throw the complement $\{(m,n) \in \mathbb{N} \times \mathbb{N} \,|\, m < n\}$ into waste basket $\{0\} \subset \mathbb{N}$. But this is not a good interpretation of *truncated* (!) subtraction: Value 0 is *not* waste, it has an important meaning as zero.

"The" waste basket $\{\perp\}$ should be an entity with a *natural* extra representation, and we should have only one such entity in a later theory of defined partial PR maps to come. This theory, to be called **PRa**$\mathbb{X}$, will be constructed with the help of *Universal Object* $\mathbb{X}$ which is to contain *codes* of all singletons and (nested) pairs of natural numbers, and "below" these codes it has room for code of *undefined value* symbol $\underline{\perp}$, in a "Hilbert's hotel".

(ii) A **PR**-map $f : A \to B$ such that f "is" a **PRa**-map

$f : \{A \,|\, \chi \vee \chi' : A \to 2\} \to \{B \,|\, \psi\},$ also "works" as a **PRa**-map
$f : \{A \,|\, \chi\} \to \{B \,|\, \psi\},$ and a **PRa**-map
$g : \{A \,|\, \chi\} \to \{B \,|\, \psi \wedge \psi'\}$ also "works" as a **PRa**-map
$g : \{A \,|\, \chi\} \to \{B \,|\, \psi\}.$

Since map-properties of *injectivity, epi-property* of **PR**-maps viewed as **PRa**-maps, **depend** on choice of hosting (predicative) **PRa** Objects—**examples** above—*specification* of a **PRa** map $f : \{A \,|\, \chi\} \to \{B \,|\, \psi\}$ must contain, besides **PR**-map $f : A \to B$, Domain and Codomain *Objects* $\chi : A \to \mathbb{2}$ and $\psi : B \to \mathbb{2}$ as well.

This way the members of <u>Map</u> <u>set</u> <u>family</u> $\mathbf{PRa}(A, B) :\ A, B$ **PRa**-Objects, become mutually <u>disjoint</u>. Inclusions $i : A' \xrightarrow{\subseteq} A''$ are realised in **PRa** as restricted **PR**-identities $\mathrm{id}_A : \{A \,|\, \chi'\} \xrightarrow{\subseteq} \{A \,|\, \chi''\},\ \chi' \implies \chi''$.

Chapter 2

Partial PR Maps

Intuitive background for present approach is the notion of a *partial PR map* in *general,* not just *defined* partial in the sense of maps within Theories $\mathbf{PRa}, \mathbf{PRa}\mathbb{X} \sqsupset \mathbf{PR}$ above.

In a first section of present PART B we arrive at a **Structure Theorem** for *Partial-map Extension* $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$, **proved** in a second section.

Here theory $\mathbf{S}$ is taken as a strengthening of Theory $\mathbf{PRa}$ or $\mathbf{PRa}\mathbb{X}$. This includes Theory $\mathbf{PRa}$ itself as well as *Descent PR Theory* $\pi\mathbf{R}$ to be studied in PART C.

Such theory $\widehat{\mathbf{S}}$ turns out to still have the structure of a *diagonal monoidal* PR Theory, and has theory $\mathbf{S}$ embedded as such a theory Essentially these two assertions make up our **Structure Theorem** for theories $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$.

Cartesian Structure is lost in part, since the—still present—(terminal map and) projection families do not preserve their character as *natural* *transformations* in the extension, see BUDACH & HOEHNKE 1975.

This type of partial-map Extension turns out to be a **Closure Operator,** $\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}$: partial *partial* maps are—have a representation within $\widehat{\widehat{\mathbf{S}}}$ as—just *partial maps,* out of $\widehat{\mathbf{S}}$.

Within variable-free, categorical theories $\widehat{\mathbf{S}}$ of *partial* PR maps, we discuss *category based* μ-recursion as well as *content driven loops,* here: while loops. This prepares in particular discussion of *termination* for suitable special such loops, namely those given as *Complexity Controlled Iterations,* for which iteration *step* decreases a *complexity* measure within a suitably given (constructive) *ordinal* O, "until" minimum 0 of O is reached. Central example for such O is $O := \mathbb{N}[\omega] \equiv \mathbb{N}^+ \subset \mathbb{N}^*$ with its lexicographic order.

[Strictly speaking, the material of present PART B is not necessary for (decisive) PART C with its self-consistency result for Descent theory $\pi\mathbf{R}$. But it relates the notion of (correct) *termination* of a *Complexity Controlled Iteration* used therein to the usual one known for μ-recursive maps given in the form of while loops.]

2.1 Theories of Partial PR Maps

We start **structure discussion** of partial map extension $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$ with a map theoretic, categorical **Definition** of *partial* $\mathbf{S}$ maps, theory $\mathbf{S}$ a strengthening of Theory $\mathbf{PRa} = \mathbf{PR} + \text{(abstr)}$ of Primitive Recursion with predicate abstraction, or of Universe PR Theories $\mathbf{PR}\mathbb{X}, \mathbf{PRa}\mathbb{X}$ of last section of PART A.

These partial PR maps are introduced as (external) pairs

$$f = \langle d_f : D_f \to A,\ \widehat{f} : D_f \to B \rangle : A \rightharpoonup B,$$

forming a *hook* type *diagram,* the "hook" in

$$\begin{tikzcd} D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & \\ A \arrow[r, harpoon, "f"'] & B \end{tikzcd}$$

defining the horizontal *partial arrow* $f : A \rightharpoonup B$ as a **partial PR map,** cf. Johnstone 1977 who admits just monic maps in the left component.

In that (special) case, D_f becomes—up to iso—a *subobject* of A, at least, if the Topos considered admits special, "standard" equaliser maps $e : A' \to A$, and such *standard* equalisers are considered to define *inclusions* within the Topos—or first order Topos—in question. This is the case for **set** theory.

In general here, "total" PR map $d_f : D_f \to A$ stands for enumeration of the *arguments* for which $f : A \rightharpoonup B$ is considered to be *defined,* and $\widehat{f} : D_f \to B$ for the (total) PR map taken as *action* or *"rule"* to be applied to the (enumeration indices of the) *defined* elements.

The **problem** in present context is that we want to **define** general *μ-recursive* (partial) maps (just) as *partial PR* maps in the sense above.

For this end we (must?) rely on the above **definition,** since in general the subset of *defined arguments*—as subset of its (wider) *Domain*—is only PR *enumerated,* not PR *decided:* We can—and must be able—to **define** such a μ-recursive map just by enumerating its *defined-arguments* and by giving a (PR) rule operating on “these” arguments.

To these “defined” arguments we have PR *access* only (?) via the enumeration *indices* which “live” in D_f.

Obviously we have to make sure—by **definition** of the notion of a *partial* **S** map below—that the PR **rule,** $\widehat{f} : D_f \to B$, maps indices pointing to the same argument, to the same *value* within Codomain B.

If we wanted to admit as *defined-argument enumeration* only *monos* resp.*inclusions,* we would need, for each map $f : A \to B$, a factorisation—**(Co) Homomorphism Theorem**—

$$\begin{tikzcd} \mathrm{KP}\, f \arrow[r, shift left] \arrow[r, shift right] & A \arrow[rr, "f"] \arrow[dr, "\mathrm{nat\,KP}\, f"'] & & B \\ & & \mathrm{CoIm}\, f = A/\mathrm{KP}\, f \arrow[ur, "\mathrm{coim}\, f"'] & \end{tikzcd}$$

with $\mathrm{KP}\, f = \{(a', a'') \in A^2 \mid f(a') \doteq_A f(a'')\} \overset{\ell}{\underset{r}{\rightrightarrows}} A$ **Kernel Pair** of $f : A \to B$.

Quotient $\mathrm{CoIm}\, f = A/\mathrm{KP}\, f$ is a quotient by a **relation,** not available so far, and hard to construct “over” **PRa**, “into” a suitable conservative enrichment, cf.Reiter 1980.

On the other hand, factorisation of an (arbitrary) **PRa**-map $f : A \to B$, even for f a *mono*—factorisation of form

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & \mathrm{Im}[f] = \mathrm{Eq}[\mathrm{im}[f], \mathrm{true}_B] \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

would need an *image predicate*

$$\mathrm{im}[f] = \mathrm{im}[f](b) : B \to \mathbb{2},$$

with its universal *minimum* property, classically **defined** by

$$\mathrm{im}[f](b) = \begin{cases} \text{true} \equiv s\,0 \text{ if } (\exists a \in A)\, f(a) \doteq b, \\ \text{false} \equiv 0 \text{ if } (\forall a \in A) \neg\, [\, f(a) \doteq b\,]. \end{cases}$$

For f mono, $\mathrm{im}[f] : B \to \mathbb{2}$ would be just the *characteristic function* of f in the sense of the **Elementary Theory of Topoi.** In Topos Theory context, **quantification** is—can be—obtained from availability of **characteristic functions,** of **monos:** But in the present context, we try to *eliminate,* better: to *avoid*—a priori—such "full" quantification, which has been relied on, by Ehrig & Kühnel & Pfender 1975, for *equational characterisation* of μ-recursion.

These **Comments** given, the next target is to prepare, to state—and then to **prove**—a **Structure Theorem** for theory (theories) $\widehat{\mathbf{S}}$ of *partial* **S** *maps:*

$\widehat{\mathbf{S}}$ becomes a diagonal monoidal Primitive Recursive Theory, with **S** naturally embedded via

$$\mathbf{S} \ni \langle f : A \to B \rangle \mapsto \langle \mathrm{id} : A \to A,\ f : A \to B \rangle \in \widehat{\mathbf{S}}$$

as a diagonal monoidal subtheory.

At first view, you may "canonically" <u>interprete</u> all of the material of this section and the one to follow, consistently into ("within") Theory **PA** of Peano Arithmetic, or into **ZF**.

Theory **PA** can be viewed as theory $\widehat{\mathbf{PA}}$ of *partial* **PA** maps, having embedded the (Cartesian, PR) Theory **PA** of *total* **PA**-maps, both enriched with—schema of abstraction—formal extensions $\{a \in A \,|\, \chi(a)\}$ of PR predicates $\chi = \chi(a) : A \to \mathbb{2}$, A an Object of **PR**, i. e.a finite (bracketed) power of $\mathbb{N}$.

As sketched above, we **define** a *partial* **S** map (an $\widehat{\mathbf{S}}$ "map" or $\widehat{\mathbf{S}}$ "morphism" or "partial") as an (external) pair $f = \langle d_f : D_f \to A\,,\, \widehat{f} : D_f \to B \rangle : A \rightharpoonup B$ with *well-defined, unique values:*

For f *defined* on $a \in A$, i. e.a "of form" $a = d_f(\hat{a})$—for general PR d_f this is *not* PR decidable—the partial map f is meant to be defined on a by $f(a) = \widehat{f}(d_f(\hat{a}))$.

The pair $f = \langle d_f\,,\, \widehat{f} \rangle$ is to fullfill the **right-uniqueness condition**

$$d_f(\hat{a}) \doteq_A d_f(\hat{a}') \implies \widehat{f}(\hat{a}) \doteq_B \widehat{f}(\hat{a}') :$$

f is to be *well-defined* on its *defined* arguments.

We now **define Theory** (theories) $\widehat{\mathbf{S}}$ of *partial* **S**-maps $f : A \rightharpoonup B$ as follows:

Objects of $\widehat{\mathbf{S}}$ are to be those of **S** :

$$\mathbb{1}, \mathbb{N}, \ldots, (A \times B), \ldots, \{A \,|\, \chi\}, \ldots$$

As *morphisms* of $\widehat{\mathbf{S}}$—the *partial* **S**-maps $f : A \rightharpoonup B$—we take those which are (externally) PR enumerated by use of the following

Partial-Map Schema:

$$\gamma f = \gamma f(\hat{a}) : D_f \to A \times B \ \ \mathbf{S}\text{-map},$$

called *graph* (of $f : A \rightharpoonup B$ to be introduced),

$$d_f = d_f(\hat{a}) \ =_{\text{def}} \ \ell \circ \gamma f : D_f \to A$$

defined arguments enumeration

$$\widehat{f} = \widehat{f}(\hat{a}) : D_f \to B \ \textit{rule}$$

$$\mathbf{S} \vdash \ d_f(\hat{a}) \doteq d_f(\hat{a}') \implies \widehat{f}(\hat{a}) \doteq \widehat{f}(\hat{a}') : D_f^2 \to \mathbb{2}$$

right uniqueness

$$(\widehat{\mathbf{S}}) \qquad \overline{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$

$$f \ =_{\text{def}} \ \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$$

$\widehat{\mathbf{S}}$-morphism, *partial* $\mathbf{S}$ map, “partial”.

Comment: We here took the *graph* $\gamma f : D_d \to A \times B$ of f to formally define f as a partial $\mathbf{S}$ map. The alternative—see above—is to define f as the (external) pair

$$f = \langle d_f : D_f \to A \,, \, \widehat{f} : D_f \to B \rangle : A \rightharpoonup B.$$

In the present *Cartesian case* the two definitions are equivalent: they transform easily into each other, by Fourman's uniqueness equations for the induced into a (Cartesian) product, and by Godement's equations for the induced into a—Cartesian—product, here for $(d_f, \widehat{f}) : D_f \to A \times B$.

But later we want to show extension by partiality to be a *Closure* operator (on suitable theories): $\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}$: “partial *partial* maps equal—

within $\widehat{\widehat{\mathbf{S}}}$—suitable (embedded) $\widehat{\mathbf{S}}$-maps." We will profit for this from the *graph*-definition of partial maps, this time over $\widehat{\mathbf{S}}$, since *then,* for a given $\widehat{\widehat{\mathbf{S}}}$ map

$$f = \langle \gamma f : D_f \rightharpoonup A \times B \rangle = \langle (d_f, \widehat{f}) : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B$$

automatically, for the defined-arguments enumerations

$$d_{d_f} : D_{d_f} \rightharpoonup D_f, \quad d_{\widehat{f}} : D_{\widehat{f}} \rightharpoonup D_f, \text{ we get:}$$
$$D_{d_f} = D_{\widehat{f}} = D_{\gamma f}.$$

This coincidence would *not* be given in case for arbitrary (external) $\widehat{\mathbf{S}}$ pairs

$$\langle d_{d_f} : D_f \rightharpoonup D_f \,, d_{\widehat{f}} : D_{\widehat{f}} \rightharpoonup D_f \rangle.$$

Problem: Can *Free-Variables* calculus for $\mathbf{S}$ be extended to the theory $\widehat{\mathbf{S}}$ of partial $\mathbf{S}$ maps?

Intuitively, use of a so-to-say "Free Variable $a \in A$", in $f = f \mathbin{\widehat{\circ}} (a) : A \rightharpoonup B$ can be <u>interpreted</u> as $f = f \mathbin{\widehat{\circ}} (a) = \widehat{f}(\hat{a})$ with $\widehat{\circ}$ the composition of $\widehat{\mathbf{S}}$, and substitution $a := d_f(\hat{a})$, $\hat{a} \in D_f$ Free Variable.

Nothing is said for the "undefined case" $f(a)$, a not of form $a \doteq d_f(\hat{a})$.

Similarly for the paired-arguments case $g : A \times B \rightharpoonup C$:

$$g(a,b) \ =_{\text{def}} \ g \mathbin{\widehat{\circ}} (\ell_{A,B}, r_{A,B}) \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} \mathrm{id}_{A \times B} \mathrel{\widehat{=}} g : A \times B \rightharpoonup C,$$

$\widehat{=}$ being theory $\widehat{\mathbf{S}}$'s "undefined", *basic* notion of equality *between*—partial—maps.

The **problem** is that $\widehat{\mathbf{S}}$ will *not* come with a *Cartesian* product, *universal* in the sense of Godement: The Cartesian product of $\mathbf{S}$

gives $\widehat{\mathbf{S}}$ just the structure of a diagonal, symmetric *monoidal category,* with in addition projection families (in particular terminal maps). But the latter loose their properties of *natural transformations* when placed into $\widehat{\mathbf{S}}$. In particular,

$$\ell \,\widehat{\circ}\, (f, g) \ =_{\text{by def}} \ \ell \,\widehat{\circ}\, (f \times g) \,\widehat{\circ}\, \Delta_C \ \widehat{\subsetneqq} \ f$$

for $f : C \rightharpoonup A, \ g : C \rightharpoonup B$, if f's domain of definition is *not contained* in that of g, see below.

This seems to me, for the moment, to cause a danger for (general) use of Free-Variables calculus—developped for the *Cartesian* (PR) case—within a partial-map framework such as theory $\widehat{\mathbf{S}}$.

But in the present context, it is easy to avoid such danger: We will use, extensively, the "full" Free-Variables calculus only for *Cartesian* PR theory (theories) $\mathbf{S}$, this also, when embedded into (monoidal) theory $\widehat{\mathbf{S}}$, and when interpreted within strengthenings or extensions of $\widehat{\mathbf{S}}$, among them theory $\mathbf{PA}$, when viewed as theory $\widehat{\mathbf{PA}}$ of *partial* maps, containing its theory of *total* maps, $\mathbf{PA}$, embedded.

Equality of partial maps: Given $f', f \ : \ A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, and $i : D_{f'} \to D_f$ (in $\mathbf{S}$) such that

$$\begin{aligned} d_{f'} &=^{\mathbf{S}} d_f \circ i : D_{f'} \to D_f \to A, \ \text{as well as} \\ \widehat{f'} &=^{\mathbf{S}} \widehat{f} \circ i : D_{f'} \to D_f \to B, \end{aligned}$$

we say that f *extends* f' or that f' is a *restriction* of f, written $f' \,\widehat{\subseteq}\, f : A \rightharpoonup B$. In diagram form:

$$
\begin{tikzcd}
 & D_{f'} \arrow[d, "i"] \arrow[ddl, bend right, "d_{f'}"'] \arrow[ddr, bend left, "\widehat{f'}"] & \\
=^{\mathbf{S}} & D_f \arrow[dl, "d_f"'] \arrow[dr, "\widehat{f}"] & =^{\mathbf{S}} \\
A \arrow[rr, harpoon, "f'", "f"'] & & B
\end{tikzcd}
$$

subgraph DIAGRAM: $f' \mathrel{\widehat{\subseteq}} f$

The schema

$$
(\ \widehat{=}\ ^{\mathbf{S}}) \quad \frac{f \mathrel{\widehat{\subseteq}} g, \ g \mathrel{\widehat{\subseteq}} f : A \rightharpoonup B}{f \mathrel{\widehat{=}} g : A \rightharpoonup B}
$$

of *equality by equal extension* is to (externally PR) define Theorie's $\widehat{\mathbf{S}}$'s notion of equality $\widehat{=}\ ^{\mathbf{S}}$, denoted $\widehat{=}$ for short.

Notation: From now on, $f = g : A \to B$ will always denote equality between maps within theory $\mathbf{S}$ choosen as *basic,* Cartesian PR theory, strengthening of $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$. Equality between *partial* $\mathbf{S}$-maps, $\widehat{\mathbf{S}}$-morphisms $f, \ g : A \rightharpoonup B$, "then" is noted $f \mathrel{\widehat{=}} g : A \rightharpoonup B$, see the above. "Pointed" equality $\dot{=} : \mathbb{N}^2 \to 2$ resp. $\dot{=}_A : A^2 \to 2$ is reserved for equality *predicates* (special *maps*), *on* $\mathbb{N}$ resp. *on* Objects A of $\mathbf{S}$, i. e.on Objects A of $\mathbf{PRa}$.

Remark: We have replaced the requirement of $d_f : D_f \to A$ to be a *monic* map in the Topos theoretic definition of *partial maps,* cf.JOHNSTONE 1977, by *right uniqueness* of the *relation, graph* $f = \gamma f = (d_f, \widehat{f}) : D_f \to A \times B$; this choice has been taken, since the here inherent concept of a **coimage** $A \xrightarrow{\text{nat}} A/\mathrm{KP}\, f$ —factorisation of Domain A of a map $f : A \to B$ by its **Kernel Pair** $\mathrm{KP}\, f \rightrightarrows A$ —, can be defined *constructively,* by enumerating a suitable *notion of equality,* while definition of an image *predicate* $\mathrm{im}[f] : B \to \mathbb{2}$ for **each** map $f : A \to B$ would need $\mathbb{2} \cong \mathbb{1} + \mathbb{1}$ to become a *subObject classifier,* i. e.availability of—non constructive—*(full) existential quantification.*

Definition of Composition and Identies for extension $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$:

Composition $h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$ of $\widehat{\mathbf{S}}$ maps

$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$ and
$g = \langle (d_g, \widehat{g}) : D_g \to B \times C \rangle : B \rightharpoonup C$

is **defined** by using an *inverse image* construction available via *pullback* in $\mathbf{S}$, by next diagram below.

[Our inverse image *is* just a pullback, pullbacks are obtained via binary (Cartesian) products and equalisers, equalisers as special extensions via equality *predicate,* and $\mathbf{S}$ has the latter by construction.]

$$\begin{tikzcd}
D_h \arrow[d, "\pi_\ell"] \arrow[dr, "\pi_r"] \arrow[ddrr, bend left, "\widehat{h}"] \arrow[dd, bend right=60, "d_h"'] \arrow[dd, phantom, bend right=30, "="] & & \\
D_f \arrow[r, phantom, "\text{p.b.}"] \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & D_g \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] \arrow[r, phantom, "="] & {} \\
A \arrow[r, "f"] \arrow[rr, bend right, "\widehat{=}", "h = g \mathbin{\widehat{\circ}} f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Composition DIAGRAM for $\widehat{\mathbf{S}}$

[The idea is from BRINKMANN-PUPPE 1969: They construct composition of *relations* this way via pullback]

Remark: The *standard form* of the pullback D_h is

$$D_h = \{(\hat{a}, \hat{b}) \in D_f \times D_g \mid \widehat{f}(\hat{a}) \doteq_B d_g(\hat{b})\},$$

with pullback-*projections*

$$\begin{aligned}
&\ell = \pi_\ell = \ell \circ \subseteq : D_h \to D_f \times D_g \to D_f \text{ and} \\
&r = \pi_r = r \circ \subseteq : D_h \to D_f \times D_g \to D_g.
\end{aligned}$$

[We may abbreviate such *restricted* projections—*pullback "projections"*—π_ℓ and π_r respectively, by ℓ, r—as suggested above]

In a sense, the pullback D_h represents the inverse image $D_h = \overset{-1}{f}[D_g]$, more precisely: $[D_h \xrightarrow{\ell} D_f] = \overset{-1}{\widehat{f}}[D_g \xrightarrow{d_g} B]$. But the definability domains d_f, d_g, d_h need not to be monic (injective).

Composition $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$ gives a *well-defined* partial map h, since for $(\hat{a}, \hat{b}), (\hat{a'}, \hat{b'}) \in D_h$ free:

$$\begin{aligned}
& d_h(\hat{a}, \hat{b}) \doteq_A d_h(\hat{a}', \hat{b}') \iff d_f(\hat{a}) \doteq_A d_f(\hat{a}') \\
& \quad \implies \widehat{f}(d_f(\hat{a})) \doteq_B \widehat{f}(d_f(\hat{a}')) \ (f \text{ well-defined}), \text{ and further} \\
& \quad \implies d_g(r(\hat{a}, \hat{b})) \doteq_B d_g(r(\hat{a}', \hat{b}')) \ (\ (\hat{a}, \hat{b}), (\hat{a'}, \hat{b'}) \in D_h, \ \text{p.b. commutes}) \\
& \quad \iff d_g(\hat{b}) \doteq_B d_g(\hat{b}') \implies \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') \\
& \quad \implies \widehat{h}(\hat{a}, \hat{b}) = \widehat{g}(\hat{b}) \doteq_C \widehat{g}(\hat{b}') = \widehat{h}(\hat{a}, \hat{b}') : D_h \times D_h \to \mathbb{2},
\end{aligned}$$

and this is well-definedness of composition $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$.

Obviously, $\widehat{\mathbf{S}}$-map $\mathrm{id}_A^{\widehat{\mathbf{S}}} =_{\mathrm{def}} \langle (\mathrm{id}_A, \mathrm{id}_A) : A \to A^2 \rangle : A \rightharpoonup A$ works as *identity* for Object A with respect to composition $\widehat{\circ}$ for $\widehat{\mathbf{S}}$ just introduced.

If one of two $\widehat{\mathbf{S}}$ maps to be composed, is an (embedded) $\mathbf{S}$ map, $\widehat{\mathbf{S}}$ *composition* becomes simpler:

Mixed Composition Lemma:

(i) For $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, and $g : B \to C$ in $\mathbf{S}$:

$$g \mathbin{\widehat{\circ}} f = \langle (d_f, g \circ \widehat{f}) : D_f \to A \times C \rangle : A \rightharpoonup C,$$

in DIAGRAM form:

$$\begin{tikzcd}
D_f \arrow[d, "d_f"'] \arrow[dr, "\widehat{f}"'] \arrow[drr, bend left, "g \circ \widehat{f}"] & & \\
A \arrow[r, harpoon, "f"] \arrow[rr, harpoon, bend right, "g \mathbin{\widehat{\circ}} f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

(ii) For $f : A \to B$ in $\mathbf{S}$, $g : B \rightharpoonup C$ in $\widehat{\mathbf{S}}$:

$$g \,\widehat{\circ}\, f = \langle (\overset{-1}{f}[d_g], \widehat{g} \circ \bar{f}) : \overset{-1}{f}[D_g] \to A \times C \rangle : A \rightharpoonup C,$$

as DIAGRAM:

$$\begin{tikzcd}
\overset{-1}{f}[D_g] \arrow[r, "\bar{f}"] \arrow[d, "{\overset{-1}{f}[d_g]}"'] \arrow[dr, phantom, "\text{p.b.}"] & D_g \arrow[d, "d_g"'] \arrow[dr, "\widehat{g}"] & \\
A \arrow[r, "f"] \arrow[rr, bend right, "g \,\widehat{\circ}\, f"'] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Proof: Immediate, since *pullbacks* pull back *identities* **q.e.d.**
We now state—**Proof** next **chapter**—

Structure Theorem for $\widehat{\mathbf{S}}$:

(i) $\widehat{\mathbf{S}}$ carries a—canonical—structure of a *diagonal symmetric monoidal category*, with *partial* composition $\widehat{\circ}$ and identities introduced above, (monoidal) product $\times$ extending $\times$ of $\mathbf{S}$, *association* ass : $(A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$, *symmetry, "transposition"* Θ : $A \times B \xrightarrow{\cong} B \times A$, and *diagonal* $\Delta : A \to A \times A$ inherited from $\mathbf{S}$; cf.BUDACH & HOEHNKE 1975 and—later[1]—PFENDER 1974 for an axiomatic description of categories of *partial maps* as monoidal categories with suitable substitution families, here including only *half-terminal* maps and *half-projective* ones in the terminology of BUDACH & HOEHNKE, since the latter $\mathbf{S}$ families are no longer *natural transformations* for Theory $\widehat{\mathbf{S}}$.

[1]there is an earlier preprint by BUDACH & HOEHNKE

(ii) The **defining diagram** for a $\widehat{\mathbf{S}}$-map—namely

$$\begin{tikzcd} D_f \arrow[d, "d_f"] \arrow[dr, bend left, "\widehat{f}"] & \\ A \arrow[r, harpoon, "f"] & B \end{tikzcd}$$

Partial Map DIAGRAM

—constitutes in fact a **commuting $\widehat{\mathbf{S}}$ diagram.**

Conversely—same notation as above—**define** the **minimised opposite**—the formally *partial,* $\widehat{\mathbf{S}}$ map

$$d_f^- = \langle (d_f, [\ ]_{\widehat{f}}) : D_f \to A \times D_f \rangle : A \rightharpoonup D_f,$$

as *opposite (graph)* to given $\mathbf{S}$ map $d_f : D_f \to A$, made *right-unique* by **selecting** D_f*-minimal* $\widehat{f}$ *equivalence representant*

$$[\ ]_{\widehat{f}} = [\alpha]_{\widehat{f}} =_{\text{def}} \min_{D_f}\{\alpha' \leq \alpha \mid \widehat{f}(\alpha') \doteq_B \widehat{f}(\alpha)\} : D_f \to D_f,$$

minimal with respect to here *"canonical",* CANTOR-ordering on $\mathbf{S}$ Object $D_f = \{D \mid \delta : D \to \mathbb{2}\}$ *inherited* from "its" *"mother"* fundamental Object D, say, this object in turn (well) ordered via canonical *counting*

$$\text{cantor}_D = \text{cantor}_D(n) : \mathbb{N} \xrightarrow{\cong} D,$$

(see general schema above of *PR dominated minimum*), and **get** the commuting $\widehat{\mathbf{S}}$-DIAGRAM

$$\begin{tikzcd}
D_f \arrow[rd, bend left, "\widehat{f}"] & \\
A \arrow[u, "d_f^-"] \arrow[r, harpoon, "f"] \arrow[ur, phantom, "\widehat{=}"] & B
\end{tikzcd}$$

put together:

$$\begin{tikzcd}
D_f \arrow[d, shift right, "d_f"'] \arrow[rd, bend left, "\widehat{f}"] & \\
A \arrow[u, shift right, "d_f^-"'] \arrow[r, harpoon, "f"] \arrow[ur, phantom, "\widehat{=}"] & B
\end{tikzcd}$$

Basic Partial Map DIAGRAM

(iii) **"section lemma:"** The first factor $f : A \rightharpoonup B$ in an $\widehat{\mathbf{S}}$ composition

$$h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C,$$

when giving an (embedded) $\mathbf{S}$ map $h : A \to C$, is itself an (embedded) $\mathbf{S}$ map:

a first composition factor of a total map is total.

So each **section** ("coretraction") of theory $\widehat{\mathbf{S}}$ is an $\mathbf{S}$ map, in particular an $\widehat{\mathbf{S}}$ section of an $\mathbf{S}$ map belongs to $\mathbf{S}$.

(iv) $\widehat{\mathbf{S}}$ clearly inherits from $\mathbf{S}$ FOURMAN's *uniqueness equation:* For $h : C \rightharpoonup A \times B$ in $\widehat{\mathbf{S}}$:

$$h \,\widehat{=}\, (h \,\widehat{\circ}\, \ell, h \,\widehat{\circ}\, r) : C \rightharpoonup A \times B,$$

where for $f : C \rightharpoonup A$, $g : C \rightharpoonup B$,

$$(f, g) \ =_{\mathrm{def}} \ (f \times g) \mathbin{\widehat{\circ}} \Delta_C : C \to C \times C \rightharpoonup A \times B,$$

with *diagonal* $\Delta_C : C \to C \times C$ of $\mathbf{S}$.

This equation guarantees *uniqueness* of the "*induced*" $(f, g) : C \rightharpoonup A \times B$, but (f, g) does not satisfy (both of) the *Cartesian equations*

$$\ell \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} f \text{ and } r \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} g,$$

except f and g have *equal domains of definition*, i. e.if $i : D_f \to D_g$, $j : D_g \to D_f$ are available such that $d_g \circ\ i = d_f : D_f \to A$ as well as $d_f \circ\ j = d_g : D_g \to A$.

(v) *Iteration* $f^{\S} : A \times \mathbb{N} \rightharpoonup A$ of $\widehat{\mathbf{S}}$-endo is available in $\widehat{\mathbf{S}}$, satisfying the characteristic $\widehat{\mathbf{S}}$-equations

$$f^{\S} \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \ =_{\text{by def}} \ f^{\S} \mathbin{\widehat{\circ}} (A \times 0) \circ \Delta_A \mathrel{\widehat{=}} \mathrm{id}_A : A \to A, \text{ and}$$
$$f^{\S} \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} f \mathbin{\widehat{\circ}} f^{\S} : A \times \mathbb{N} \rightharpoonup A.$$

(vi) Freyd's uniqueness of the *initialised iterated* holds in $\widehat{\mathbf{S}}$:

$$(\mathrm{FR!})_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} f : A \rightharpoonup B, \ g : B \rightharpoonup B, \ h : A \times \mathbb{N} \rightharpoonup B \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ h \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \rightharpoonup f : A \rightharpoonup B \ \text{ and} \\ h \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} h : A \times \mathbb{N} \rightharpoonup B \end{array}}{h \mathrel{\widehat{=}} g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) : A \times \mathbb{N} \rightharpoonup B \times \mathbb{N} \rightharpoonup B.}$$

[The latter two statements are not so easy to prove: PR *construction* of *comparison* maps is needed, for comparing the respective enumerations of defined arguments in the postcedents, proceeding from the comparison maps given by the antecedents]

(vii) For extension $\widehat{\mathbf{S}}$ of $\mathbf{S}$ again, we get—by the general FREYD's argument—the corresponding **full schema of primitive recursion,** namely

$$(\mathrm{pr})_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}} \text{ (initialisation)}, \\ h : (A \times \mathbb{N}) \times B \rightharpoonup B \text{ (step map)} \end{array}}{\begin{array}{l} f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \rightharpoonup B \text{ is available in } \widehat{\mathbf{S}}, \\ \text{characterised (up to equality } \widehat{=}\text{) in } \widehat{\mathbf{S}} \text{ by} \\ f \,\widehat{\circ}\, (\mathrm{id}_A, 0) \,\widehat{=}\, g : A \rightharpoonup B \text{ and} \\ f \,\widehat{\circ}\, (A \times s) \,\widehat{=}\, h \,\widehat{\circ}\, (\mathrm{id}_{A \times \mathbb{N}}, f) \\ =_{\text{by def}} \; h \,\widehat{\circ}\, ((A \times \mathbb{N}) \times f) \,\widehat{\circ}\, \Delta_{A \times \mathbb{N}} : \\ A \times \mathbb{N} \to (A \times \mathbb{N})^2 \rightharpoonup (A \times \mathbb{N}) \times B \rightharpoonup B. \end{array}}$$

Proof of the Theorem is long, already since we have to show that many assertions, but mainly since assertion (v) needs some auxiliary arguments. We give this proof in next **chapter.**

2.2 Proof of Structure Theorem for Partials

Notation: As agreed earlier, **S** will still denote a theory **strengthening** Theory **PRa** = **PR** + (abstr) $\sqsupset$ **PR**, the *fundamental (Free-Variables) categorical Theory* **PR** of Primitive Recursion, definitionally enriched by extensions $\{A \,|\, \chi : A \to \mathbb{2}\}$ of its predicates, or of Universe PR theories $\mathbf{PR}\mathbb{X}, \mathbf{PRa}\mathbb{X}$ out of last chapter of PART A, or—classically—restriction $\mathbf{PA} \restriction \mathrm{PRa}$ of **PA** to its PRa Object- and map-terms, with all *map equations* between **PRa**-maps inherited from **PA**.

Next Chapter will focus on particular example(s) for such theories **S**, namely on strengthening(s) $\pi_O\mathbf{R} = \mathbf{PRa} + (\pi_O)$ of Theory **PRa**, by schema (π_O) of *non-infinite descent in Ordinal O, O* extending $\mathbb{N}[\infty]$.

The statement hardest to **prove** of the **Structure Theorem** will be Freyd's uniqueness schema $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$ for (diagonal monoidal) theory $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$.

For **proving** that schema, we will use—and therefore **prove** first—the following schema for **S**—of independent interest:

Schema of $\mathbb{N}$-valued Descending Iteration:

A Cartesian PR Theory **S** strengthening Theory **PRa** admits the following Uniqueness Schema:

$$\begin{array}{ll} & c = c\,(a) : A \to \mathbb{N} \text{ (argument complexity)} \\ & f = f(a) : \{a \in A \,|\, c\,(a) \doteq 0\} \to B \text{ (anchor map)} \\ & p = p(a) : A \to A \text{ (predecessor map)} \\ & \qquad \text{satisfying—“as such” predecessor} \\ & \mathbf{S} \vdash \ [\,c\,(a) > 0 \implies c\,p(a) \doteq c\,(a) - 1\,] : A \to \mathbb{2} \\ & \qquad \text{the latter alternatively, more general:} \\ & \mathbf{S} \vdash \ [\,c\,(a) > 0 \implies c\,p(a) < c\,(a)\,] : A \to \mathbb{2} \\ (\mathrm{PR} \downarrow \mathbb{N}) & \overline{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}} \\ & h = h(a) : A \to B \text{ in } \mathbf{S} \text{ (is given and) unique s. t.} \\ & \mathbf{S} \vdash \ [\,c\,(a) \doteq 0 \implies h(a) \doteq_B f(a)\,] : A \to \mathbb{2}^2 \to \mathbb{2}, \\ & \qquad \text{(initial implication), as well as} \\ & \mathbf{S} \vdash \ [\,c\,(a) > 0 \implies h(a) \doteq_B h\,p(a)\,] : A \to \mathbb{2}^2 \to \mathbb{2}, \\ & \qquad \text{(step implication).} \end{array}$$

[Equality predicate $[\,b \doteq_B b'\,] : B^2 \to \mathbb{2}$ is needed. The schema works in any Cartesian PR theory **T**, for which the Codomain B of anchor $f : A \to B$ has equality *predicate* giving *Equality Definability* for map pairs with that Codomain]

Proof of the **Schema:** $h = h(a) \ =_{\mathrm{def}} \ f\,p^{\S}(a, c\,(a)) : A \to B$ does the job, and uniquely so, since, by *Bottom up Resolution Lemma for Primitive Recursion*—PART A—the *bottom up formula for iteration,* namely

$$p^{\S}(a, s\,n) = p^{\S}(p(a), n) : A \times \mathbb{N} \to A,$$

is equivalent—provided $p(a, 0) = a : A \to A$—to the one given in the **definition** of the iterated $p^{\S}$, namely $p^{\S}(a, s\,n) = p\,p^{\S}(a, n)$.

The alternative—general—descent version "$c\,(a) > 0 \implies c\,p(a) < c\,(a)$" clearly reduces to the (direct) predecessor one, since downwards iteration then just remains possibly—for the then redundant number of steps—on argument $\bar{a}$, with $c\,(\bar{a}) \doteq 0$ reached in less than $c\,(a)$ *predecessor steps* p **q.e.d.**

This Schema given, we now start with the **Proof** of the **Structure Theorem** for theory (theories) $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$:

Proof of assertion (i): We first give to $\mathbf{S}$ the structure of a diagonal monoidal category and **verify** the defining properties of this structure:

Composition $\widehat{\circ}$ introduced above—by pullback—is **compatible** with $\widehat{\subseteq}$, and hence also with $\widehat{=}$, since for $f' \mathrel{\widehat{\subseteq}} f : A \rightharpoonup B$ and $g' \mathrel{\widehat{\subseteq}} g : B \rightharpoonup C$, we are given "inclusions" $i : D_{f'} \to D_f$ and $j : D_{g'} \to D_g$ such that for $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$ and $h' = g' \mathbin{\widehat{\circ}} f' : A \rightharpoonup B \rightharpoonup C$ **compatibility** DIAGRAM below commutes, with (unique) $k : D_{h'} \to D_h$ in $\mathbf{S}$, *induced into the pullback* D_h by $i \circ \ell' : D_{h'} \to D_{f'} \to D_f$ and $j \circ r' : D_{h'} \to D_{g'} \to D_g$.

This shows: for $f' \mathrel{\widehat{\subseteq}} f : A \rightharpoonup B,\ g' \mathrel{\widehat{\subseteq}} g : B \rightharpoonup C$:

$$h' := g' \circ f' \mathrel{\widehat{\subseteq}} g \circ f =: h : A \rightharpoonup B \rightharpoonup C.$$

Here the standard form of isomorphism $k : D'_h \to D_h$ is given by

$$D'_h \ni (\hat{a}', \hat{b}') \mapsto (i \circ \ell'(\hat{a}'), j \circ r'(\hat{b}')) \in D_h :$$

Definition by *Free-Variables Diagram Chase,* which can be replaced by unique commutative fill in, k, into the following **S**-diagram, based on the *universal properties* of Cartesian product, pullback, and—later—of the Natural Numbers Object $\mathbb{N} = \langle \mathbb{N}, 0, s \rangle$:

$$
\begin{tikzcd}
 & & & D_{h'} \arrow[dl, "\ell'"'] \arrow[dr, "r'"] \arrow[ddll, bend right, "d_{h'}"'] \arrow[ddrr, bend left, "\widehat{h'}"] \arrow[dddd, dashed, bend right=90, looseness=1.5, "k"'] & & \\
 & & D_{f'} \arrow[dl, "d_{f'}"'] \arrow[dr, "\widehat{f'}"] \arrow[dd, dotted, "i"] & \text{p.b.} & D_{g'} \arrow[dl, "d_{g'}"'] \arrow[dr, "\widehat{g'}"] \arrow[dd, dotted, "j"] & \\
 & A \arrow[rr, "f", "f'"'] \arrow[rrrr, bend right=15, "h", "h'"'] & & B \arrow[rr, "g", "g'"'] & & C \\
 & & D_f \arrow[ul, "d_f"] \arrow[ur, "\widehat{f}"'] & \text{p.b.} & D_g \arrow[ul, "d_g"] \arrow[ur, "\widehat{g}"'] & \\
 & & & D_h \arrow[ul, "\ell"] \arrow[ur, "r"'] \arrow[uull, bend left, "d_h"] \arrow[uurr, bend right, "\widehat{h}"'] & &
\end{tikzcd}
$$

Compatibility DIAGRAM[a] of $\widehat{\circ}$ with $\subseteq$

[a] F. Herrmann

For proving **associativity** of (partial) composition $\widehat{\circ}$, consider

$$
\begin{tikzcd}
D_{(h\widehat{\circ}g)\widehat{\circ}f} \arrow[dr, leftrightarrow, "\cong"] \arrow[drr] \arrow[dddr, bend right] & & & & \\
& D_{h\widehat{\circ}(g\widehat{\circ}f)} \arrow[d] \arrow[ddrr] \arrow[r, phantom, "\text{p.b.}", near start, shift left=4] & D_{h\widehat{\circ}g} \arrow[dd] \arrow[ddr] & & \\
& D_{g\widehat{\circ}f} \arrow[d] \arrow[dr] \arrow[r, phantom, "\text{p.b.}"] & {} \arrow[r, phantom, "\text{p.b.}"] & {} & \\
& D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] \arrow[r, phantom, "\text{p.b.}"] & D_g \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] & D_h \arrow[d, "d_h"] \arrow[dr, "\widehat{h}"] & \\
& A \arrow[r, rightharpoondown, "f"] & B \arrow[r, rightharpoondown, "g"] & C \arrow[r, rightharpoondown, "h"] & D
\end{tikzcd}
$$

Associativity DIAGRAM for $\widehat{\circ}$— via *nested pullbacks*

Here the standard form of isomorphism $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$ is restriction of *association isomorphism*

$$\text{ass} : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$$

to a map $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$; it is verified straightforward that $((\hat{a}, \hat{b}), \hat{c}) \in D_{(h\circ g)\circ f}$ in fact "is mapped" into $(\hat{a}, (\hat{b}, \hat{c})) \in D_{h\circ(g\circ f)}$.

Cylindrification, defined "componentwise", by

$$
\begin{aligned}
& f = \langle\langle d_f : D_f \to A\,,\ \widehat{f} : D_f \to B\rangle : A \rightharpoondown B\rangle \quad \mapsto \quad \langle C \times f\rangle \\
& \quad =_{\text{def}} \ \langle C \times d_f : C \times D_f \to C \times A\ ,\ C \times \widehat{f} : C \times D_f \to C \times B\rangle : \\
& \qquad C \times A \rightharpoondown C \times B
\end{aligned}
$$

preserves **inclusion** $f' \mathrel{\widehat{\subseteq}} f : A \rightharpoondown B$, given by $i : D'_f \to D_f$, since

$$C \times i : D_{C\times f'} = C \times D_{f'} \to C \times D_f = D_{C\times f}$$

gives the inclusion $C \times f' \mathbin{\widehat{\subseteq}} C \times f : C \times A \rightharpoonup C \times B$. Hence in particular, cylindrification preserverves (partial) **equality** $f' \mathbin{\widehat{=}} f$ defined by $f' \mathbin{\widehat{\subseteq}} f$ **and** $f \mathbin{\widehat{\supseteq}} f'$ being given simultaneously.

As for $\mathbf{S}$, the **product** of maps, given by composition of cylindrifications, namely

$$(\times_{\widehat{\mathbf{S}}}) \quad \frac{f : A \rightharpoonup A',\ g : B \rightharpoonup B' \text{ in } \widehat{\mathbf{S}}}{\begin{aligned} (f \times g) \ &=_{\text{def}} \ (f \times B') \mathbin{\widehat{\circ}} (A \times g) : \\ &\qquad A \times B \rightharpoonup A \times B' \rightharpoonup A' \times B' \\ &\mathbin{\widehat{=}} (A' \times g) \mathbin{\widehat{\circ}} (f \times B) : \\ &\qquad A \times B \rightharpoonup A' \times B \rightharpoonup A' \times B' \end{aligned}}$$

extends the *Cartesian* product of $\mathbf{S}$ into a *BiFunctor* again, on theory $\widehat{\mathbf{S}}$. It looses its universal property, essentially since $[\,\Pi_A : A \to \mathbb{1}\,]_{A \in \mathbf{Obj_S}}$ looses *naturality,* within $\widehat{\mathbf{S}}$.

Proof of **BiFunctoriality** of $\times$ in $\widehat{\mathbf{S}}$:

The main point here is **Functoriality** of **Cylindrification,** namely

$$\langle g : B \to B' \rangle \mapsto \langle A \times g : A \times B \to A \times B' \rangle :$$

For partial maps $\langle (d_g, \widehat{g}) : D_g \to B \times B' \rangle : B \rightharpoonup B'$ and $\langle (d_{g'}, \widehat{g'}) : D'_g \to B' \times B'' \rangle : B' \rightharpoonup B''$, and a ("cylindrifying") Object A, **recall** the following defining $\mathbf{S}/\widehat{\mathbf{S}}$ DIAGRAM for g, g', and $h := g' \mathbin{\widehat{\circ}} g$:

$$\begin{tikzcd}
D_h \arrow[d, "\pi_\ell"] \arrow[dr, "\pi_r"] \arrow[drr, bend left, "\widehat{h}"] \arrow[dd, bend right, "d_h"'] & & \\
D_f \arrow[r, phantom, "\text{p.b.}"] \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] & D_{g'} \arrow[d, "d_{g'}"] \arrow[dr, "\widehat{g'}"] \arrow[r, phantom, "="] & {} \\
B \arrow[r, "g"] \arrow[rr, bend right, "\widehat{=}", "h = g' \,\widehat{\circ}\, g"'] & B' \arrow[r, "g'"] & B''
\end{tikzcd}$$

Functorial—and pullback preserving—cylindrification, with Object A, inside **S**, leads to:

$$\begin{tikzcd}
A \times D_h \arrow[d, "A\times\pi_\ell"] \arrow[dr, "A\times\pi_r"] \arrow[drr, bend left, "A\times\widehat{h}"] \arrow[dd, bend right, "A\times d_h"'] & & \\
A \times D_g \arrow[r, phantom, "\text{p.b.}"] \arrow[d, "A\times d_g", "="'] \arrow[dr, "A\times\widehat{g}"] & A \times D_{g'} \arrow[d, "A\times d_{g'}"] \arrow[dr, "A\times\widehat{g'}"] \arrow[r, phantom, "="] & {} \\
A \times B \arrow[r, "A\times g"] \arrow[rr, bend right, "\widehat{=}", "A\times h"'] & A \times B' \arrow[r, "A\times g'"] & A \times B''
\end{tikzcd}$$

Functoriality DIAGRAM for Theory $\widehat{\mathbf{S}}$

Here—by **definition** of the **cylindrified**—

$$d_{A\times h} \;=_{\text{def}}\; A \times d_h : A \times D_h \to A \times B, \text{ and}$$
$$\widehat{A \times h} \;=_{\text{def}}\; A \times \widehat{h} : A \times D_h \to A \times B''.$$

Hence the *frame* DIAGRAM in the above just **defines** partial PR map $A \times h : A \times B \rightharpoonup A \times B''$.

From this—commuting—“Functoriality DIAGRAM”, with its pullback obtained by—pullback preserving—cylindrification from the earlier one, we get

$$\begin{aligned} d_{A\times h} \ &=_{\text{by def}}\ A \times d_h = (A \times d_g) \circ (A \times \pi_\ell) \\ &\qquad \text{by functoriality of } \times \text{ in } \mathbf{S}, \\ &= d_{(A\times g')\,\widehat{\circ}\,(A\times g)}. \end{aligned}$$

The latter by pullback preservation of *cylindrification* $A\times \_ : \mathbf{S} \longrightarrow \mathbf{S}$.

This shows the *Domain enumeration* property of (monoidal) functoriality of $A \times \_ : \widehat{\mathbf{S}} \longrightarrow \widehat{\mathbf{S}}$.

Rule property: Again from this DIAGRAM, we get

$$\begin{aligned} A \times \widehat{h} &= A \times (\widehat{g' \,\widehat{\circ}\, g}) \\ &= (A \times \widehat{g'}) \circ (A \times \pi_r) \\ &=_{\text{by def}}\ (\widehat{A \times g'})\,\widehat{\circ}\,(\widehat{A \times g}), \end{aligned}$$

again by $\times$ functorial in $\mathbf{S}$, and cylindrification preserving Cartesian products and pullbacks, within $\mathbf{S}$.

This **proves** functoriality of cylindrification in $\widehat{\mathbf{S}}$.

[A more elegant, “global” argument says: Both $A\times D_h$ and $D_{(A\times g')\,\widehat{\circ}\,(A\times g)}$ are *projective Limits* of the lower-two-rows part of the DIAGRAM, when coming with their respective *cones.* Therefore they admit a “comparing” *natural isomorphism,* and that’s what is needed for proving the wanted functoriality of cylindrification within Theory $\widehat{\mathbf{S}}$]

$\widehat{\mathbf{S}}$ inherits from $\mathbf{S}$ **transposition**

$$\Theta = \Theta_{A,B}(a,b) \ =_{\text{def}}\ (b,a) = (r,\ell) : A \times B \xrightarrow{\cong} B \times A$$

as well as **diagonal**

$$\Delta = \Delta_A(a) \ =_{\text{def}} \ (a, a) = (\text{id}, \text{id}) : A \to A \times A,$$

and **association**

$$\begin{aligned} \text{ass} = &\text{ass}_{A,B,C}((a, b), c) \ =_{\text{def}} \ (a, (b, c)) = (\ell\ell, (r\ell, r)) : \\ &((A \times B) \times C) \xrightarrow{\cong} (A \times (B \times C)). \end{aligned}$$

It is obvious that $\widehat{\mathbf{S}}$ inherits *naturality* of the *transformation* families ass, Θ, and Δ.

Using these natural transformations, we get—out of functoriality of cylindrification—in particular *BiFunctoriality* of (binary) *product* $\times$ within Theory $\widehat{\mathbf{S}}$. This shows assertion (i) of the **Structure Theorem.**

For **Proof** of first half of assertion (ii), namely

$$f \mathbin{\widehat{\circ}} d_f \mathrel{\widehat{=}} \widehat{f} : A \rightharpoonup B$$

for given partial

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B,$$

consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ diagram:

$$\begin{tikzcd}
\mathrm{KP}[d_f] \arrow[r, "r"] \arrow[d, "\ell"] \arrow[dr, phantom, "\text{p.b.}"] & D_f \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & \\
D_f \arrow[r, "d_f"] \arrow[u, bend left, "\Delta"] \arrow[rr, bend right, "\widehat{f}"'] & A \arrow[r, harpoon, "f"] & B
\end{tikzcd}$$

Partial Map Definition DIAGRAM

This $\mathbf{S}/\widehat{\mathbf{S}}$ diagram shows on one hand

$$f \mathbin{\widehat{\circ}} d_f = \langle (\ell, \widehat{f} \circ r) : \mathrm{KP}[\, d_f \,] \to D_f \times B \rangle : D_f \rightharpoonup B$$

$$\mathbin{\widehat{\subseteq}}\ \widehat{f} = \langle (\mathrm{id}_{D_f}, \widehat{f}) \rangle : D_f \rightharpoonup B,$$

via $\ell : \mathrm{KP}[d_f] \subseteq D_f^2 \xrightarrow{\ell} D_f$,

with $\widehat{f}$ embedded as its graph $(\mathrm{id}_{D_f}, \widehat{f})$,

see obvious definition of the embedding below.

The opposite (graph) inclusion, via $\Delta : D_f \to \mathrm{KP}[\, d_f \,]$ (given by reflexivity of *kernel pair* $\mathrm{KP}[\, d_f \,]$), namely

$$\widehat{f} = \langle (\mathrm{id}_{D_f}, \widehat{f}) \rangle \mathbin{\widehat{\subseteq}} f \mathbin{\widehat{\circ}} d_f$$

is immediate from commutativity of $\mathbf{S}$ diagram above with—formally—partial arrow $f : A \rightharpoonup B$ removed.

For **Proof** of second $\widehat{\mathbf{S}}$ equality of assertion (ii), consider the following DIAGRAM, which commutes in its $\mathbf{S}$ part (frame):

$$\begin{tikzcd}
D_f \arrow[d, "d_f"] \arrow[dr, "{[\ ]_{\widehat{f}}}"] \arrow[dd, bend right=60, leftrightarrow, "\mathrm{id}"'] & & \\
A \arrow[r, harpoon, "d_f^-"] & D_f \arrow[r, "\widehat{f}"] & B \\
D_f \arrow[u, "d_f"] \arrow[urr, "\widehat{f}"'] & &
\end{tikzcd}$$

In fact,

$$\begin{aligned}
&\widehat{f} \circ [\ ]_{\widehat{f}}(\hat{a}) \ =_{\mathrm{def}}\ \widehat{f} \circ [\ ]_{\mathrm{KP}[\widehat{f}]}(\hat{a}) \\
&\quad =_{\text{by def}}\ \widehat{f}\,(\min_{D_f}\{\hat{a}' \le \hat{a} \mid \widehat{f}(\hat{a}') \ \dot{=}_B\ \widehat{f}(\hat{a})\}) = \widehat{f}(\hat{a}) : D_f \to B,
\end{aligned}$$

by **definition** of *choice of minimal representant* $[\ ]_\rho : D_f \to D_f$ for *equivalence predicate*

$$\rho := \mathrm{KP}[\widehat{f}] \xrightarrow{\subseteq} D_f^2.$$

This establishes present second half of assertion (ii). Together, we have shown all three $\widehat{\mathbf{S}}$ commutativities of assertion (ii).

Proof of (iii): For $f : A \rightharpoonup B,\ g : B \rightharpoonup C$ given in the assertion, consider—with the usual notation for defined-arguments enumerations and rules—the following DIAGRAM, showing their "total" composition $h = \langle (\mathrm{id}_A, h) : A \to A \times C \rangle : A \to C$. This DIAGRAM just enriches Composition DIAGRAM in forgoing chapter by the data of h and comparison $\mathbf{S}$ map $j : A \to D_{g \widehat{\circ} f}$ which establishes "graph inclusion" $h \mathrel{\widehat{\subseteq}} g \widehat{\circ} f : A \rightharpoonup C$.

$$\begin{tikzcd}
D_{g \widehat{\circ} f} \arrow[d, "\pi_\ell"] \arrow[dr, "\pi_r"] \arrow[ddrr, bend left, "\widehat{g \widehat{\circ} f}"] & & \\
D_f \arrow[r, phantom, "\text{p.b.}"] \arrow[d, "d_f"] \arrow[dr, "\widehat{f}"] & D_g \arrow[r, phantom, "="] \arrow[d, "d_g"] \arrow[dr, "\widehat{g}"] & {} \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
A \arrow[u, "\mathrm{id}_A"] \arrow[urr, "h"] \arrow[uuu, bend left, "j"] & &
\end{tikzcd}$$

composition-total DIAGRAM for $\widehat{\mathbf{S}}$

Now define $k := \pi_\ell \circ j : A \to D_{g \widehat{\circ} f} \to D_f$, having section property $d_f \circ k = \mathrm{id}_A : A \rightarrowtail D_f \twoheadrightarrow A$ inherited from comparison property of j :

$A \rightarrowtail D_{g \,\widehat{\circ}\, f}$. This gives the assertion, for embedded $\widetilde{f} : A \to B$, taken as **S** representant of $f : A \rightharpoonup B$, with $\widetilde{f} \;=_{\mathrm{def}}\; \widehat{f} \circ k : A \rightarrowtail D_f \to B$.

In fact,

$$\begin{aligned} \widetilde{f} &= \widehat{f} \circ k \mathrel{\widehat{=}} (f \,\widehat{\circ}\, d_f) \,\widehat{\circ}\, k, \\ &\qquad \text{this by first part of forgoing assertion} \\ &\mathrel{\widehat{=}} f \,\widehat{\circ}\, \mathrm{id}_A \text{ (section property of } k = \pi_\ell \circ j \text{)} \\ &\mathrel{\widehat{=}} f : A \rightharpoonup B \text{ as required.} \end{aligned}$$

Proof of (v): **Iteration** extends to partial endomaps

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times A \rangle : A \rightharpoonup A :$$

Define the domain of definition $d_{f^\S} : D_{f^\S} \to A \times \mathbb{N}$ of the (wanted) partial PR map $f^\S : A \times \mathbb{N} \rightharpoonup A$ PR—in **S**—as follows:

$$\begin{aligned} D_{f^\S} \;=_{\mathrm{def}}\; \{\alpha' = \alpha; \langle a \rangle = \langle \hat{a}_n; \ldots; \hat{a}_1; a \rangle \in D_f^*; \langle A \rangle \mid \\ \widehat{f}(\hat{a}_1) \doteq a \wedge \bigwedge_{j=2}^{n-1} \widehat{f}(\hat{a}_j) \doteq d_f(\hat{a}_{j-1})\}, \end{aligned}$$

as Free-Variables diagram chase, with $a_j := d_f(\hat{a}_j)$:

$$\begin{tikzcd} \hat{a}_n \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \hat{a}_{n-1} \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \cdots & \hat{a}_2 \arrow[dr, mapsto, "\widehat{f}"] & \hat{a}_1 \arrow[d, mapsto, "d_f"] \arrow[dr, mapsto, "\widehat{f}"] & \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a \end{tikzcd}$$

Intuitively this means $(a_n, n) \overset{f^\S}{\mapsto} a \in A$, via the f-*defined arguments* $a_j \;\doteq_A\; d_f(\hat{a}_j)$:

$$A \ni a_n \overset{f}{\mapsto} a_{n-1} \overset{f}{\mapsto} \ldots \overset{f}{\mapsto} a_1 \overset{f}{\mapsto} a \in A.$$

Formally,

$$\hat{a}_j =_{\text{by def}} \pi(j, \alpha) : D_f^* \smallsetminus \{\Box\} \to D_f,$$

via suitably PR defined *string projection "family"* π.

For the anchor case $n = 0$, $\alpha = \Box \in D_f^*$ is the empty string, and $\alpha' = \langle a \rangle \in \langle A \rangle \cong A$.

It is clear intuitively that we further (must) **define**, PR on $n = \text{length}(\alpha)$, $\alpha' = \mathit{cat}(\alpha, \langle a \rangle) \in D_{f^\S}$ within **S**, the *components* $d_{f^\S} = d_{f^\S}(\alpha') : D_{f^\S} \to A \times \mathbb{N}$, as well as $\widehat{f^\S} = \widehat{f^\S}(\alpha) : D_{f^\S} \to A$, as follows:

$$d_{f^\S}\,\langle a \rangle =_{\text{def}} (a, 0) : D_{f^\S} \supset \langle A \rangle \to A \times \mathbb{N},$$

$$(\text{“iteration length” } \text{length}(\alpha) = 0),$$

$$d_{f^\S}\,\langle \hat{a}_n; \ldots; \hat{a}_1; a \rangle =_{\text{def}} (d_f(\hat{a}_n), n) \text{ for } n = \text{length}(\alpha) \geq 1.$$

as well as

$$\widehat{f^\S}(\alpha') = \widehat{f^\S}(\mathit{cat}(\alpha, a)) =_{\text{def}} a : D_{f^\S} \to A.$$

It is clear now that our zig-zag chain of applying rule $\widehat{f}$ of f and (successfully?) "searching" for a defining index—$\hat{a}_j$—for once "earlier" applying $\widehat{f}$, defines "the" right *partial map* $f^\S : A \times \mathbb{N} \rightharpoonup A$, and that therefore

$$f^\S = \langle (d_{f^\S}, \widehat{f^\S}) : D_{f^\S} \to (A \times \mathbb{N}) \times A \rangle : A \times \mathbb{N} \rightharpoonup A$$

constructed above, fullfills the equations for an iterated of f within theory $\widehat{\mathbf{S}}$; detailed **proof** by Peano Induction on *iteration length* $n = \text{length}(\alpha)$ above, proof within Cartesian PR theory **S**.

Proof of (vi): Freyd's uniqueness (FR!) for theory $\widehat{\mathbf{S}}$, namely of **uniqueness** of the **initialised iterated** for that theory, needs some preparation:

Assertion (vi) is proved—from (iv) and (v)—by the arguments for Freyd's Theorem—which work mutatis mutandis also for present case $\widehat{\mathbf{S}}$: the morphisms are constructed PR within $\widehat{\mathbf{S}}$ as in case of theory $\mathbf{S}$. The *characteristic* equations $\widehat{=}$ for these specific constructs are proven in parallel to case $\mathbf{S}$, and uniqueness is **proved** using Freyd's uniqueness $(\mathrm{FR!})_{\mathbf{S}}$ for theory $\mathbf{S}$, as well as by use of FOURMAN's uniqueness of *"induced"* $\widehat{\mathbf{S}}$-morphisms into a product $A \times B$, in case they are "available": and this will here be the case.

Now, for proving assertion (vi), on Freyd's uniqueness $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$—of the *initialised iterated* for theory $\widehat{\mathbf{S}}$—we first show, use of proven statements (i) to (v)—the following

Endo Driven Uniqueness for $\widehat{\mathbf{S}}$:

$$
(\mathrm{Uni})_{\widehat{\mathbf{S}}} \quad
\frac{
\begin{array}{l}
f = \langle (d_f, \widehat{f}) : D_f \to (A \times \mathbb{N}) \times B \rangle : A \times \mathbb{N} \rightharpoonup B, \\
g = \langle (d_g, \widehat{g}) : D_g \to (A \times \mathbb{N}) \times B \rangle : A \times \mathbb{N} \rightharpoonup B \text{ in } \widehat{\mathbf{S}}, \\
h : B \rightharpoonup B \text{ in } \widehat{\mathbf{S}} \text{ such that:} \\
f \,\widehat{\circ}\, (id_A, 0) \,\widehat{=}\, g \,\widehat{\circ}\, (id_A, 0) : A \to A \times \mathbb{N} \rightharpoonup B, \\
\qquad \text{(anchor equality)}, \\
f \,\widehat{\circ}\, (A \times s) \,\widehat{=}\, h \,\widehat{\circ}\, f, \text{ (step equality), as well as} \\
g \,\widehat{\circ}\, (A \times s) \,\widehat{=}\, h \,\widehat{\circ}\, g : A \times \mathbb{N} \to A \times \mathbb{N} \rightharpoonup B : \\
\qquad \text{(same induction step comportment)}
\end{array}
}{
f \,\widehat{=}\, g : A \times \mathbb{N} \rightharpoonup B.
}
$$

Proof of this uniqueness by PR *construction* of *comparison maps*

$$D_f \underset{j}{\overset{i}{\rightleftarrows}} D_g$$

compatible with $d_f : D_f \to A \times \mathbb{N}$ and $d_g : D_g \to A \times \mathbb{N}$ on one hand, and with $\widehat{f} : D_f \to B$, $\widehat{g} : D_g \to B$ on the other hand.

By symmetry of the assertion, we need to construct just comparison map $i : D_f \to D_g$ (within **S**). This will be done "PR" on *dependent parameter* $n := r\, d_f : D_f \to A \times \mathbb{N} \to \mathbb{N}$.

[PR Availability of such a—PR—construction, within **S**, is given by **Schema of $\mathbb{N}$-Valued Descending Iteration** which has been **proven** at begin of chapter]

For anchoring **S**-map $D_f \overset{i}{\dashrightarrow} D_g$, we **define**—as a possible form of pullback in the diagram below—

$$\begin{aligned} D_{f\,\widehat{\circ}\,(\mathrm{id}_A,0)} = D_f^0 \ &=_{\mathrm{def}} \ \{\alpha \in D_f \mid d_f(\alpha) \doteq (a,0)\} \\ &=_{\mathrm{by\,def}} \ \{\alpha \in D_f \mid r\, d_f(\alpha) \doteq 0\} \subseteq D_f, \\ &\text{with } a := \ell\, d_f(\alpha) : D_f \to A \times \mathbb{N} \to A. \end{aligned}$$

Analogeously for $g : A \times \mathbb{N} \rightharpoonup B$ in place of f.

Now consider $\mathbf{S}/\widehat{\mathbf{S}}$ diagram

$$\begin{tikzcd}
D_f^0 \arrow[r, "\subseteq"] \arrow[d, "\bar{d}_f"] \arrow[dd, bend right, "i_0"'] \arrow[dr, phantom, "\text{p.b.}"] & D_f \arrow[d, "d_f"] \arrow[dr, bend left, "\widehat{f}"] \arrow[dd, dashed, bend right, "i"'] & \\
A \arrow[r, "(\mathrm{id}_A{,}0)"] \arrow[dr, phantom, "\text{p.b.}"] & A \times \mathbb{N} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
D_g^0 \arrow[u, "\bar{d}_g"'] \arrow[r, "\subseteq"] & A \times \mathbb{N} \arrow[u, "d_g"'] \arrow[ur, bend right, "\widehat{g}"'] &
\end{tikzcd}$$

Anchor DIAGRAM for $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$

Here comparison

$$i_0 : D_f \supseteq D_f^0 = D_{f \,\widehat{\circ}\, (id,A)} \to D_g^0 \subseteq D_g,$$

(just) making the ($\mathbf{S}$ part of) diagram above commute, is given by the *anchoring antecedent* of our assertion; analogeously: $j_0 : D_g^0 \to D_f^0$.

Similarly we argue for the PR construction step: Assume $i = i(\alpha) : D_f \to D_g$ *known* and *compatible* for $\alpha \in D_f$ such that $d_f(\alpha) \doteq (a, n) \in A \times \mathbb{N}$, i. e.for α such that $r\, d_f(\alpha) \doteq n$.

Then for defining $i(\alpha') \in D_g$ for $d_f(\alpha') \doteq (a', s\,n)$, i. e.$r\, d_f(\alpha') \doteq s\,n$, consider the $\mathbf{S}/\widehat{\mathbf{S}}$ diagram below, with

$$\begin{aligned} D_{f \,\widehat{\circ}\, (A \times s)} &=_{\text{by def}} D_f^+ =_{\text{def}} \{\alpha' \in D_f \mid d_f(\alpha) \doteq (a', s\,n) \in A \times \mathbb{N}_{\geq 1}\} \\ &= \{\alpha' \in D_f \mid d_f(\alpha') \geq 1\}, \text{ and } h = \langle (d_h, \widehat{h}) : D_h \to B \times B \rangle : B \rightharpoonup B \end{aligned}$$

out of the antecedent.

This is a quasi-canonical choice for $D_{f \,\widehat{\circ}\, (A \times s)}$, namely the *inverse image* of "inclusion" $(A \times s) : A \times \mathbb{N} \rightarrowtail A \times \mathbb{N}$.

$$
\begin{tikzcd}
 & & D_f \arrow[d, "d_f"] \arrow[drr, bend left, "\widehat{f}"] & & \\
D_{f\,\widehat{\circ}\,(A\times s)} \arrow[r, equal] & D_f^+ \arrow[ur, "\subseteq"] \arrow[d, "d_f\restriction"] \arrow[dd, phantom, ""] \arrow[r, phantom, "\text{p.b.}"] \arrow[ddd, bend right=60, "i_f^+"'] & A\times\mathbb{N} \arrow[drr, "f"] & & \\
 & A\times\mathbb{N} \arrow[ur, "A\times s"'] \arrow[dr, "f"] & & \widehat{=} & B \\
 & D_f \arrow[u, "d_f"] \arrow[r, "\widehat{f}"'] & B \arrow[urr, "h"] & & \\
 & D_{h\,\widehat{\circ}\,f} \arrow[u, "\bar{d_h}"] \arrow[r, "\widehat{\widehat{f}}"'] \arrow[ur, phantom, "\text{p.b.}"] \arrow[uuu, bend left=40, "j_f^+"] & D_h \arrow[u, "d_h"] \arrow[uurr, bend right, "\widehat{h}"'] & &
\end{tikzcd}
$$

Step antecedent DIAGRAM for f in $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$

Imagine the **parallel** step diagram for $g : A \times \mathbb{N} \rightharpoonup B$ to lie *under* this step diagram for f. Both layers commute in their $\mathbf{S}$ part: this is what the **step part** of our **antecedent** says, namely $\widehat{=}$ commutativities of the central diamonds (for f resp.g), via suitable $\mathbf{S}$ maps $i_f^+ : D_f^+ \to D_{h\,\widehat{\circ}\,f}$ and—reverse "inclusion"— $j_f^+ : D_{h\,\widehat{\circ}\,f} \to D_f^+$; same for g—"lower layer"—with $i_g^+ : D_g^+ \to D_{h\,\widehat{\circ}\,g}$ and $j_g^+ : D_{h\,\widehat{\circ}\,g} \to D_g^+$ given.

Suppose now—**recursion hypothesis**—$\mathbf{S}$-map $i = i(\alpha) : D_f \to D_g$ to be (PR) *constructed* "for all" $\alpha \in D_f$ "such that" $d_f(\alpha) \doteq (a, n)$, i. e.with $r\, d_f(\alpha) \doteq n : D_f \to A \times \mathbb{N} \xrightarrow{r} \mathbb{N}$.

This given, we want to define—PR—$i(\alpha') \in D_g$ for $d_f(\alpha') = (a', s\, n)$, i. e.for $\alpha' \in D_f$ with $r\, d_f(\alpha') \doteq s\, n \in \mathbb{N}$.

We do this by Free-Variables diagram chase as follows, looking at the two layers of step antecedent DIAGRAM above, starting with

$$\alpha' \in D_f^+ \ =_{\text{by def}} \ \{\alpha' \in D_f \,|\, r\, d_f(\alpha') \geq 1\} :$$

By step antecedent $f \,\widehat{\circ}\, (A \times s) \,\widehat{=}\, h \,\widehat{\circ}\, f \,:\, A \times \mathbb{N} \rightharpoonup B$ we are given **S**-maps

$$i_f^+ : D_f^+ \to D_{h \,\widehat{\circ}\, f} = \{(\alpha, \hat{b}) \in D_f \times D_h \,|\, d_f(\alpha) \ \doteq_B \ d_h(\hat{b})\},$$

(canonical form of $D_{h \,\widehat{\circ}\, f}$), and j_f^+ in the other direction, the two making the **S** part of the diagram commute.

So we are lead to **define**—by Free-Variables diagram chase, starting with $\alpha' \in D_f$ satisfying $d_f(\alpha') \doteq (a', s\, n) \in A \times \mathbb{N}_{\geq 1}$, and hence with $\alpha' \in D_f^+ = D_{f \,\widehat{\circ}\, (A \times \mathbb{N})}$—see diagram above:

$$D_f^+ \ni \alpha' \overset{i_f^+}{\mapsto} i_f^+(\alpha') =: (\alpha, \hat{b}) \in D_{h \,\widehat{\circ}\, f} \subseteq D_f \times D_h$$

—in the (canonical) pullback $D_{h \,\widehat{\circ}\, f,}$ see above—, and further:

$$\begin{aligned} & D_{h \,\widehat{\circ}\, f} \ni i_f^+(\alpha') = (\alpha, \hat{b}) \\ & \mapsto (i(\alpha), \hat{b}) \in D_{h \,\widehat{\circ}\, g} \ [\, \subseteq D_g \times D_h \,] \\ & \mapsto i(\alpha') \ =_{\text{def}} \ i(\alpha), \end{aligned}$$

i. e.recursively:

$$i(\alpha') \ =_{\text{def}} \ i(\alpha) \ =_{\text{by def}} \ i \,\ell\, i_f^+(\alpha') : D_f \supseteq D_f^+ \to D_f \to D_g.$$

Here $d_g\, i(\alpha) \ \doteq_{A \times \mathbb{N}} \ d_f(\alpha)$, as well as $\widehat{g}\, i(\alpha) \ \doteq_B \ \widehat{f}(\alpha)$, the latter three construction and assertions given by **recursion hypothesis** on n. These properties carry over – by recursive construction of $i(\alpha')$—from $\alpha \in D_f$ with $d_f(\alpha) \doteq (a, n)$—to $\alpha' \in D_f$ with $d_f(\alpha') \doteq (a', s\, n)$.

This way we have established—recursively—an inclusion $f \,\widehat{\subseteq}\, g$: $A \times \mathbb{N} \rightharpoonup B$, via $i : D_f \to D_g$.

Symmetrically to this (graph) inclusion $f \mathrel{\widehat{\subseteq}} g$, we get an inclusion $g \mathrel{\widehat{\subseteq}} f$, via $j : D_g \to D_f$ (say), and hence the wanted **Uniqueness** assertion $f \mathrel{\widehat{=}} g : A \times \mathbb{N} \rightharpoonup B$ for theory $\widehat{\mathbf{S}}$.

[For *Cartesian* PR theories $\mathbf{R}$, in particular for strengthenings $\mathbf{S}$ of **PRa**, this uniqueness holds trivially, by Freyd's uniqueness $(\mathrm{FR!})_{\mathbf{R}}$]

For recursive construction of comparison map $i : D_f \to D_g$ in the **proof** above, we have used—implicitely—the **Schema of ℕ-Valued Descending Iteration.**

Explicitely this construction of $i = i(\alpha) : D_f \to D_g$ works—based on the mentioned **schema**—as follows:

As argument *Domain* we take in present case $\tilde{A} := D_f$, as ℕ-valued *complexity*

$$\tilde{c} := r\, d_f = r\, d_f(\alpha) : D_f \to A \times \mathbb{N} \xrightarrow{r} \mathbb{N},$$

and as *predecessor* endo map $\tilde{g} = \tilde{g}(\alpha) : D_f \to D_f$, we take

$$\begin{aligned}
\tilde{g}(\alpha) &=_{\mathrm{def}} \alpha \in D_f^0 \subset D_f \text{ for } \tilde{c}(\alpha) = r\, d_f(\alpha) \doteq 0, \\
&\quad \text{i. e.for } \alpha \text{ of form } \alpha = (a, 0) \in A \times \mathbb{N}, \text{ as well as} \\
\tilde{g}(\alpha') &=_{\mathrm{def}} \ell\, i_f^+(\alpha) : D_f \supset D_f^+ \to D_{h \,\widehat{\circ}\, f} \xrightarrow{\ell} D_f \text{ for } \tilde{c}(\alpha') = r\, d_f(\alpha') > 0.
\end{aligned}$$

Then resulting $\mathbf{S}$-map $i \;=_{\mathrm{def}}\; \tilde{h} = \tilde{h}(\alpha) : D_f \to D_g$ – resulting by the above **Descent Schema** for $\mathbf{S}$ from $\tilde{c}$, $\tilde{f}$, and $\tilde{g}$—does its job: it yields in fact—now **proven** formally by use of the Descent Schema—the asserted domain-enumeration and rule *compatible graph inclusion* $i : f \mathrel{\widehat{\subseteq}} g : A \times \mathbb{N} \rightharpoonup B$. Same for inclusion in the other direction: $j : g \mathrel{\widehat{\subseteq}} f : A \times \mathbb{N} \rightharpoonup B$.

This **Endo Driven Uniqueness** for $\widehat{\mathbf{S}}$, now easily yields remaining assertion (v) of the **Structure Theorem** for $\widehat{\mathbf{S}}$, namely Freyd's uniqueness $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$ for the *initialised iterated* $h : A \times \mathbb{N} \rightharpoonup B$, as follows:
Because of

$$\begin{aligned}
& h \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \mathrel{\widehat{=}} f : A \rightharpoonup B \\
& \qquad\qquad \text{by "initial" condition for } h : A \times \mathbb{N} \rightharpoonup B \\
& \mathrel{\widehat{=}} g^{\S} \circ (f \times \mathbb{N}) \mathbin{\widehat{\circ}} (\mathrm{id}_A, 0) \ \text{(initialisation of iteration)} \\
& \qquad\qquad \text{as well as} \\
& h \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} h : A \times \mathbb{N} \rightharpoonup B \ \text{(step condition for } h\text{)} \\
& \qquad\qquad \textbf{and,} \text{ obvious:} \\
& g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) \mathbin{\widehat{\circ}} (A \times s) \mathrel{\widehat{=}} g \mathbin{\widehat{\circ}} g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) : A \times \mathbb{N} \rightharpoonup B,
\end{aligned}$$

the pair $h,\ g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) : A \times \mathbb{N} \rightharpoonup B$ fullfills the antecedent of **Endo Driven Uniqueness** for theory $\widehat{\mathbf{S}}$. From this uniqueness, we eventually get $\widehat{\mathbf{S}}$-equality

$$h \mathrel{\widehat{=}} g^{\S} \mathbin{\widehat{\circ}} (f \times \mathbb{N}) : A \times \mathbb{N} \rightharpoonup B$$

i. e.Freyd's uniqueness (FR!) for theory $\widehat{\mathbf{S}}$.

This finishes the **Proof** of the **Structure Theorem** for **Extension $\widehat{\mathbf{S}}$** of theory **S**, by **Partial Maps.**

What about **Equality Definability** in the *Partial Map Setting?* Its second version below is covered in part by the original schema for theories **T** with NNO, for maps $f,\ g : A \to \mathbb{N}$, with $\mathbb{N}$'s equality *predicate* $[\, m \doteq n \,] : \mathbb{N}^2 \to \mathbb{N}$, and the other (Codomain) Objects of Theory **PRa** inheriting their equality predicate from $\mathbb{N}$. But in this *partial map* case of theories $\widehat{\mathbf{S}}$, we get another, important version by replacing *equality* $\mathrel{\widehat{=}}$ with *overall truth* by inclusion, as follows:

Equality Definability Theorem for Theories $\widehat{\mathbf{S}}$ of Partial Maps: Theories $\widehat{\mathbf{S}}$ admit the following two **schemata** of *Equality Definability:*

$$(\mathrm{EqDef}_{\widehat{\subseteq}})\quad \frac{\begin{array}{l} f,\ g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}}, \\ [\, f \ \dot{=}_B\ g\,] \mathrel{\widehat{\subseteq}} \mathrm{true}_A : A \rightharpoonup \mathbb{2}, \text{ i. e.} \\ \dot{=}_B\ \widehat{\circ}\,(f \times g)\,\widehat{\circ}\,\Delta_A \mathrel{\widehat{\subseteq}} \mathrm{true}_A \end{array}}{f \restriction D \ =_{\text{by def}}\ f\,\widehat{\circ}\,d \mathrel{\widehat{=}} g \restriction D \ =_{\text{by def}}\ g\,\widehat{\circ}\,d : D \to A \rightharpoonup B.}$$

Here $D = D_f \cap D_g = D_f \underset{d_f, d_g}{\times} D_g$ is the pullback—diamond in the DIAGRAM below—of the defined arguments enumerations $d_f : D_f \to A$ and $d_g : D_g \to B$ (in $\mathbf{S}$).

The original schema, with stronger antecedent has a stronger postcedent then that expected *formally* from the general one for theories $\mathbf{T}$, namely:

$$(\mathrm{EqDef}_{\widehat{=}})\quad \frac{\begin{array}{l} f,\ g : A \rightharpoonup B \text{ in } \widehat{\mathbf{S}}, \\ [\, f \ \dot{=}_B\ g\,] \mathrel{\widehat{=}} \mathrm{true}_A : A \rightharpoonup \mathbb{2}, \end{array}}{\begin{array}{l} f \mathrel{\widehat{=}} \widehat{f} = \widehat{g} : A \to B \\ \mathrel{\widehat{=}} g : A \rightharpoonup B. \end{array}}$$

Under the strong condition of *equality* $\widehat{=}$ with overall *truth* of $[\,f \doteq_B g\,] : A \rightharpoonup \mathbb{2}$, f and g become $\widehat{=}$ equal—as we already know—but in addition, they then necessarily admit $\widehat{=}$ *representants* within $\mathbf{S}$, namely their *rules* $\widehat{f} : D_f \to B$ resp.$\widehat{g} : D_g \to B$. Other way round: $d_f : D_f \to A$ as well as $d_g : D_g \to B$ admit sections d_f^- resp.d_g^- – within $\mathbf{S}$—and

$$f \mathrel{\widehat{=}} \widehat{f} \circ d_f^- : A \to D_f \to B \quad \text{and} \quad g \mathrel{\widehat{=}} \widehat{g} \circ d_g^- : A \to B.$$

Proof: For both schemata consider the following $\mathbf{S}/\widehat{\mathbf{S}}$-DIAGRAM:

$$\begin{tikzcd}
 & & A \arrow[dd, "\mathrm{id}"] \arrow[ddrrr, bend left, "\mathrm{true}_A"] \arrow[ddll, dotted, bend right, "j"'] & & & \\
 & D_f \arrow[dr, "d_f"] \arrow[ddrrr, bend left, "\widehat{f}"] & & & & \\
D \arrow[ur, "\pi_\ell"] \arrow[dr, "\pi_r"'] \arrow[rr, "d"] \arrow[uurr, bend left, "i=d"] & & A \arrow[r, "\Delta"] \arrow[rr, bend right, "(f{,}g)"'] & A^2 \arrow[r, harpoon, "f\times g"] & B^2 \arrow[r, "\doteq_B"] & \mathbb{2} \\
 & D_g \arrow[ur, "d_g"] \arrow[rrr, "\widehat{g}"'] & & & B \arrow[u, "\Delta"'] \arrow[ur, "\mathrm{true}_B"'] &
\end{tikzcd}$$

Partial-Maps Equality Definability DIAGRAM

For **proof** of *inclusion* variant ($EqDef_{\widehat{\subseteq}}$) of the schema, $\mathbf{S}$-map $i : D \to A$ is to establish the (antecedent) inclusion

$$[\,f \doteq_B g\,] \mathrel{\widehat{\subseteq}} \mathrm{true}_A : A \rightharpoonup \mathbb{2} :$$

The diamond on the left is a pullback, to give *defined arguments enumeration* of

$$\doteq_B \widehat{\circ}\, (f, g) =_{\text{by def}} \doteq_B \widehat{\circ}\, (f \times g)\, \widehat{\circ}\, \Delta_A : A \to A^2 \rightharpoonup B^2 \to B,$$

i. e.of $(f \times g) \,\widehat{\circ}\, \Delta_A : A \rightharpoonup B^2$ as its (commuting) *diagonal* $d : D \to A$, intuitively as the *intersection* $d = d_{(f,g)} : D = D_{(f,g)} = D_f \cap D_g \to A$.

Necessarily then, $i = d : D \to A$, and commutativity of the **S**-part of the DIAGRAM shows

$$[\widehat{f} \circ \pi_\ell \doteq_B \widehat{g} \circ \pi_r] = \mathrm{true}_D : D \to B^2 \to \mathbb{2},$$

and hence—by **Equality Definability** for theory **S** :

$$\widehat{f} \circ \pi_\ell = \widehat{g} \circ \pi_r : D \to B.$$

But this just means

$$f \restriction D \ =_{\text{by def}} \ f \,\widehat{\circ}\, d \,\widehat{=}\, g \,\widehat{\circ}\, d \ =_{\text{by def}} \ g \restriction D : A \rightharpoonup B,$$

and this is what we wanted to **prove** in the **inclusion-into-truth** variant of the **schema.**

In the **partial-equality** variant of the schema, we have—in the DIAGRAM—additional, reverse *inclusion* $j : A \to D$, making "everything" commute, in particular $d \circ j = \mathrm{id}_A : A \to D \to A$.

From $\mathbf{S}/\widehat{\mathbf{S}}$-commutativity of the DIAGRAM, we get in particular

$$\begin{aligned}
&(f, g) \,\widehat{=}\, (f, g) \,\widehat{\circ}\, \mathrm{id}_A \,\widehat{=}\, (f, g) \,\widehat{\circ}\, d \circ j \\
&\widehat{=}\, (f \,\widehat{\circ}\, d, g \,\widehat{\circ}\, d) \,\widehat{\circ}\, j \,\widehat{=}\, (f \,\widehat{\circ}\, d_f \,\widehat{\circ}\, \pi_\ell, g \,\widehat{\circ}\, d_g \,\widehat{\circ}\, \pi_r) \,\widehat{\circ}\, j \\
&\widehat{=}\, (\widehat{f} \circ \pi_\ell, \widehat{g} \circ \pi_r) \circ j \\
&\qquad \text{the latter by (ii) of } \textit{Structure Theorem} \text{ for } \widehat{\mathbf{S}} \\
&\widehat{=}\, (\widehat{f} \times \widehat{g}) \circ (\pi_\ell \circ j, \pi_r \circ j) : \\
&A \longrightarrow D_f \times D_g \xrightarrow{\widehat{f} \times \widehat{g}} B^2.
\end{aligned}$$

This shows that from $\widehat{=}$ variant of the antecedent, in fact we get representation of

$$(f, g) \ =_{\text{by def}} \ (f \times g) \circ \Delta_A : A \to A^2 \rightharpoonup B^2$$

as a PR map—within theory **S**—and hence the same for components f and g, represented by their *rules* $\widehat{f} : D_f \to B$ and $\widehat{g} : D_g \to B$ respectively. Equality $f \,\widehat{=}\, g : A \rightharpoonup B$ then follows – using this representation as **S**-maps, by earlier schema (EqDef) of Equality Definability for theory **S** **q.e.d.**

2.3 Partial-Map Extension as Closure *

As **proved** above, theory $\widehat{\mathbf{S}}$, of *partial* **S**-maps, has (Cartesian) theory **S embedded** via

$$\langle f : A \to B \rangle \mapsto \langle (\mathrm{id}_A, f) : A \to A \times B \rangle : A \rightharpoonup B$$

as a *symmetric diagonal monoidal primitive recursive subtheory.*

Hence Theory (theories) $\widehat{\mathbf{S}}$ turn(s) out to be—in particular—**conservative** extension(s) of theory (theories) **S**.

In analogy to diagonal monoidal *extension* $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$, the extended theory $\widehat{\mathbf{S}}$ in turn admits a—diagonal monoidal—**extension** $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}}$, into a theory of partial *partial* **S**-maps, as we **show** now:

Objects of $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}} \sqsupset \mathbf{S}$ are to be—just as before—the Objects of $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$: (bracketed, finite) *powers* of $\mathbb{N}$ and *predicative* subsets $\{A \,|\, \chi\}$ of the latter.

As *morphisms* of $\widehat{\widehat{\mathbf{S}}}$, *from* Object A *to* Object B, we take, in analogy of *partial* map definition over $\mathbf{S}$, $\widehat{\mathbf{S}}$-maps of form

$f = \langle \gamma f : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B,$

where $\gamma f,$ as an $\widehat{\mathbf{S}}$ map, has (general) form

$\gamma f = \langle (d_{\gamma f}, \widehat{\gamma f}) : D_{\gamma f} \to D_f \times (A \times B) \rangle : D_f \rightharpoonup A \times B,$

with $\mathbf{S}$-maps (!) $d_{\gamma f} : D_{\gamma f} \to A$ and $\widehat{\gamma f} : D_{\gamma f} \to B.$

$d_{\gamma f} : D_{\gamma f} \to A$ and $\widehat{\gamma f} : D_{\gamma f} \to B$ are the *components* of *graph* γf of f, which in turn **defines** $f : A \rightharpoondown B$ as an $\widehat{\widehat{\mathbf{S}}}$-morphism, a partial *partial* $\mathbf{S}$-map.

For **defining** *composition* of such $\widehat{\widehat{\mathbf{S}}}$-morphisms, composition of, say, $f : A \rightharpoondown B$ and $g : B \rightharpoondown C$, consider the following $\mathbf{S}/\widehat{\mathbf{S}}/\widehat{\widehat{\mathbf{S}}}$-DIAGRAM which displays the $\widehat{\mathbf{S}}/\mathbf{S}$ data of f and g to be **composed** into an $\widehat{\widehat{\mathbf{S}}}$-morphism $g \mathbin{\widehat{\widehat{\circ}}} f : A \rightharpoondown B \rightharpoondown C$:

$$\begin{tikzcd}
D_f \arrow[d, "\gamma f"'] & D_{\gamma f} \arrow[l, "d_{\gamma f}"'] \arrow[dl, "\widehat{\gamma f}"'] & D_{g \widehat{\widehat{\circ}} f} \arrow[l, "\pi_\ell"'] \arrow[r, "\pi_r"] \arrow[dll] \arrow[drr] \arrow[d, equal] & D_{\gamma g} \arrow[r, "d_{\gamma g}"] \arrow[dr, "\widehat{\gamma g}"] & D_g \arrow[d, "\gamma g"] \\
A \times B \arrow[d, "\ell"] \arrow[drr, "r"] & & D_{\gamma f} \underset{r \circ \widehat{\gamma f},\, \ell \circ \widehat{\gamma g}}{\times} D_{\gamma g} \arrow[d, phantom, "\text{p.b.}"] & & B \times C \arrow[d, "r"] \arrow[dll, "\ell"'] \\
A \arrow[rr, "f"'] & & B \arrow[rr, "g"'] & & C
\end{tikzcd}$$

Composition DIAGRAM for $\widehat{\widehat{\mathbf{S}}}$

Composition $g \mathbin{\widehat{\widehat{\circ}}} f : A \rightharpoondown C$ then is **defined** to have as *graph* $\gamma_{g \widehat{\widehat{\circ}} f}$ the map *"induced"* by the left and right *frame* morphisms of the

DIAGRAM, namely:

$$\gamma_{g \,\widehat{\widehat{\circ}}\, f} =_{\mathrm{def}} (\ell \,\widehat{\circ}\, \gamma f \,\widehat{\circ}\, d_{\gamma f} \circ \pi_\ell \,,\, r \,\widehat{\circ}\, \gamma g \,\widehat{\circ}\, d_{\gamma g} \circ \pi_r) :$$
$$D_{g \,\widehat{\widehat{\circ}}\, f} \rightharpoonup A \times C.$$

In fact this *induced* has a representation within $\mathbf{S}$, as a *formally induced,* since

$$\gamma f \,\widehat{\circ}\, d_{\gamma f} \mathrel{\widehat{=}} \widehat{\gamma f} \quad \text{as well as} \quad \gamma g \,\widehat{\circ}\, d_{\gamma g} \mathrel{\widehat{=}} \widehat{\gamma g},$$

by assertion (ii) of the **Structure Theorem** for $\widehat{\mathbf{S}}$.

[This observation already makes it plausible that the components of an $\widehat{\widehat{\mathbf{S}}}$-map can be taken—$\widehat{\widehat{\mathbf{S}}}$-*equally*—within theory $\mathbf{S}$ itself, see below]

Since composition $\widehat{\widehat{\circ}}$ of theory $\widehat{\widehat{\mathbf{S}}}$ is defined—as already composition $\widehat{\circ}$ of $\widehat{\mathbf{S}}$—by $\mathbf{S}$-pullback, it becomes associative, "again" since (finite) limits do not change—up to natural isomorphism—when limits of "subdiagrams" are added before taking the "overall" limit.

Cylindrification for theory $\widehat{\widehat{\mathbf{S}}}$ is obvious, as are then its functor properties: just **cylindrify** each $\mathbf{S}/\widehat{\mathbf{S}}$ diagram needed. In particular, cylindrification shows up to be functorial, with respect to composition $\widehat{\widehat{\circ}}$ for $\widehat{\widehat{\mathbf{S}}}$ introduced above. That substitutions ass, Θ, Δ—embedded into $\widehat{\widehat{\mathbf{S}}}$—have their requested *substitution* properties is obvious (embedding, as earlier, see below).

We have **established** so far that theory $\widehat{\widehat{\mathbf{S}}}$ defines a **(symmetric) diagonal monoidal theory.**

Definition of the expected **embedding** $\sqsubseteq : \widehat{\mathbf{S}} \longrightarrow \widehat{\widehat{\mathbf{S}}}$ is simple:

$\widehat{\mathbf{S}}$-maps have just $\mathbf{S}$-maps as *graphs* $\gamma f = (d_f, \widehat{f}) : D_f \to A \times B$.

[Formally we always start with such a *graph* $\gamma f : D_f \to A \times B$ "as a whole"; $d_f =_{\text{by def}} \ell \circ \gamma f : D_f \to A \times B \to A$, as well as $\widehat{f} =_{\text{by def}} r \circ \gamma f : D_f \to A \times B \to B$ are "only then" **defined** as the *components* of a graph γf given]

Mapping such graph defining $\mathbf{S}$-maps, into their $\widehat{\mathbf{S}}$-versions, defines a *symmetric diagonal monoidal embedding* $\widehat{\mathbf{S}} \sqsubseteq \widehat{\widehat{\mathbf{S}}}$, in the sense of diagonal monoidal structure of $\widehat{\widehat{\mathbf{S}}} \sqsupseteq \widehat{\mathbf{S}}$ introduced above; in detail:

$$\begin{aligned} &\sqsubseteq \langle f : A \rightharpoonup B \rangle \\ &=_{\text{def}} \ \langle \langle (\mathrm{id}_A, \gamma f) = (\mathrm{id}_A, (d_f, \widehat{f})) \\ &=_{\text{by def}} \ (\mathrm{id}_A, (\ell \circ \gamma f, r \circ \gamma f)) : A \to A \times (A \times B) \rangle : \\ &\quad A \rightharpoonup (A \times B) \rangle : A \rightharpoondown B. \end{aligned}$$

In fact, this **embedding** is a **Closure,** as is already felt when regarding composition diagram above for theory $\widehat{\widehat{\mathbf{S}}}$. We now attempt to **prove** this Closure property of forming *partial partial* $\mathbf{S}$-maps.

[*Embedding* $\sqsubset : \mathbf{S} \longrightarrow \widehat{\mathbf{S}}$ was **defined** above by

$$\langle f : A \to B \rangle \mapsto \langle \langle (\mathrm{id}_A, f) : A \to A \times B \rangle : A \rightharpoonup B \rangle.$$

It constitutes a *Functor* $\sqsubset : \mathbf{S} \longrightarrow \widehat{\mathbf{S}}$, by definition of equality $f \mathrel{\widehat{=}} g : A \to B$ of $\widehat{\mathbf{S}}$-morphisms as *partial* $\widehat{\mathbf{S}}$-maps. As already said, this Embedding preserves the diagonal monoidal structure—given on $\mathbf{S}$ as a *Cartesian structure*—"into" the canonical *diagonal monoidal structure* "inherited" by $\widehat{\mathbf{S}}$ from $\mathbf{S}$. $\widehat{\mathbf{S}}$ inherits $\mathbf{S}$'s (terminal maps and) *projections* as well, but these loose their universal properties—and their character as natural transformations—within the extension]

Extension $\widehat{\widehat{\mathbf{S}}} \mathrel{\widehat{\sqsupseteq}} \widehat{\mathbf{S}}$ "again" inherits its structure as a diagonal *monoidal theory,* "directly" from theory $\mathbf{S}$: Composition for $\widehat{\widehat{\mathbf{S}}}$ is **defined** by the Closure diagram below, formally an $\mathbf{S}$-DIAGRAM, the $\widehat{\mathbf{S}}$-arrows are "inserted" for orientation.

For a **Proof** of the expected **Closure Property** $\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}$ consider this (commuting) $\widehat{\mathbf{S}}$ **Closure** diagram, for a given $\widehat{\widehat{\mathbf{S}}}$ (partial *partial* $\mathbf{S}$) map

$$f = \langle \gamma f = (\ell \mathbin{\widehat{\circ}} \gamma f,\ r \mathbin{\widehat{\circ}} \gamma f) =: (d_f, \widehat{f}) : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B.$$

[Here we have applied FOURMAN's (uniqueness) equation for the induced map which is inherited by $\widehat{\mathbf{S}}$ from $\mathbf{S}$. Starting directly with the *graph* $\gamma f : D_f \rightharpoonup A \times B$ of f, and not with its *components* $d_f,\ \widehat{f}$, is *necessary* (just) here, see discussion below]

$$\begin{tikzcd}
 & D_h \arrow[d, equal, "\mathrm{def}"] \arrow[ddd, bend right, "h=\gamma h"'] & \\
 & D_{\gamma f} \arrow[d, shift left, "d_{\gamma f}"] \arrow[dddl, bend right, "d_h:=\widehat{d_f}"'] \arrow[dd, bend left=60, "\widehat{\gamma f}"] \arrow[dddr, bend left, "\widehat{\widehat{f}}=:\widehat{h}"] & \\
 & D_f \arrow[u, shift left, "d^{-}_{\gamma f}"] \arrow[d, "\gamma f"] \arrow[ddl, "d_f"'] \arrow[ddr, "\widehat{f}"] & \\
 & A \times B \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\
A \arrow[rr, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, bend right=15, "\widehat{\widehat{=}}"] & & B
\end{tikzcd}$$

Closure DIAGRAM for Extension by partial maps

In this DIAGRAM, $\gamma f : D_f \rightharpoonup A \times B$ is the *graph* of $\widehat{\widehat{\mathbf{S}}}$-morphism $f : A \rightharpoondown B$ to be considered. The $\mathbf{S}$-maps $d_{\gamma f} : D_{\gamma f} \to D_f$ (defined-arguments enumeration) and $\widehat{\gamma f} : D_{\gamma f} \to A \times B$ (rule) are to **define** $\gamma f : D_f \rightharpoonup A \times B$ as a *partial* $\mathbf{S}$-map, an $\widehat{\mathbf{S}}$ morphism.

Graph $\gamma f : D_f \rightharpoonup A \times B$ has $\widehat{\mathbf{S}}$-*components*

$$d_f \ =_{\text{def}} \ \ell_{A,B} \,\widehat{\circ}\, \gamma f : D_f \rightharpoonup A \times B \to A \text{ and}$$

$$\widehat{f} \ =_{\text{def}} \ r_{A,B} \,\widehat{\circ}\, \gamma f : D_f \rightharpoonup A \times B \to B,$$

satisfying as such—by FOURMAN's equation for $\widehat{\mathbf{S}}$—

$$(d_f, \widehat{f}) \ =_{\text{by def}} \ (d_f \times \widehat{f}) \,\widehat{\circ}\, \Delta_{D_f} \mathrel{\widehat{=}} \gamma f : D_f \to D_f \times D_f \rightharpoonup A \times B.$$

$\widehat{\mathbf{S}}$-morphism $d^{-}_{\gamma f} : D_f \rightharpoonup D_{\gamma f}$ is the (minimised) *opposite graph* to $\mathbf{S}$ map $d_{\gamma f}$, **defined** in (ii) of the **Structure Theorem** for $\widehat{\mathbf{S}}$, satisfying as such $\widehat{\mathbf{S}}$ equation

$$\widehat{\gamma f} \,\widehat{\circ}\, d^{-}_{\gamma f} \mathrel{\widehat{=}} \gamma f : D_f \rightharpoonup D_{\gamma f} \to A \times B.$$

Choice of $\widehat{\mathbf{S}}$ representant $h : A \rightharpoonup B$ for given $\widehat{\widehat{\mathbf{S}}}$ morphism $f : A \rightharpoondown B$, h to be defined by its *graph*—with components in $\mathbf{S}$—now is simple:

Take as this representant of $f : A \rightharpoondown B$, the $\widehat{\mathbf{S}}$ map $h : A \rightharpoonup B$ given by the *frame* in the DIAGRAM above:

$$\begin{aligned} h &= \langle (d_h, \widehat{h}) : D_h \to A \times B \rangle : A \rightharpoonup B \\ &=_{\text{def}} \ \langle (\widehat{d}_f, \widehat{\widehat{f}}) : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B \\ &=_{\text{by def}} \ \langle \widehat{\gamma f} : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B. \end{aligned}$$

As particular instance of *Basic Partial Map* DIAGRAM in assertion (ii) of the **Structure Theorem** for Theory $\widehat{\mathbf{S}}$ we get the following commutative diagram:

$$\begin{tikzcd}
D_{\gamma f} \arrow[d, shift right, "d_{\gamma f}"'] \arrow[drr, bend left, "h=\widehat{\gamma f}"] \arrow[dr, phantom, "\widehat{=}"] & & \\
D_f \arrow[u, shift right, "d_{\gamma f}^{-}"'] \arrow[rr, rightharpoonup, "\gamma f"] & & A \times B
\end{tikzcd}$$

Basic Partial Map DIAGRAM for $\widehat{\mathbf{S}}$ morphism $\gamma f : D_f \rightharpoonup A \times B$

Since $\gamma f : D_f \rightharpoonup A \times B$ is the *graph* of $f : A \rightharpoondown B$ (in $\widehat{\widehat{\mathbf{S}}}$) given, this commutative $\widehat{\mathbf{S}}$ diagram shows—by **definition** of equality $\widehat{\widehat{=}}$ between $\widehat{\widehat{\mathbf{S}}}$ morphisms:

$$h \,\widehat{\widehat{=}}\, f : A \rightharpoondown B.$$

Embedding $\widehat{\mathbf{S}} \;\sqsubseteq\; \widehat{\widehat{\mathbf{S}}}$ is a diagonal monoidal functor, with—retractive—*Choice* $h : A \rightharpoonup B$ as *representant* for $f : A \rightharpoondown B$ in $\widehat{\widehat{\mathbf{S}}}$: Retractive Choice up to natural equivalence of functors. This is seen straightforward by **definition** of composition $\widehat{\widehat{\circ}}$ of $\widehat{\widehat{\mathbf{S}}}$ and of Embedding—Section $\sqsubseteq : \widehat{\mathbf{S}} \longrightarrow \widehat{\widehat{\mathbf{S}}}$.

Closure Theorem for Extension of Theory S by Partial Maps:

Closure by *Partial Maps* is **idempotent:** Partial map Closure of theory $\widehat{\mathbf{S}}$ is again a diagonal monoidal category $\widehat{\widehat{\mathbf{S}}}$ which is in fact **equivalent**—as such a category—to theory $\widehat{\mathbf{S}}$:

$$\widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}}.$$

2.4 μ-Recursion without Quantifiers

We **define** μ-Recursion within the Free-Variables framework of **partial PR maps** as follows:

Given a **PR** predicate $\varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{2}$, the $\widehat{\mathbf{S}}$ morphism

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

is to have ($\mathbf{S}$) **components**

$$\begin{aligned}
&D_{\mu\varphi} \ =_{\text{def}} \ \{A \times \mathbb{N} \,|\, \varphi\} \subseteq A \times \mathbb{N}, \\
&d_{\mu\varphi} = d_{\mu\varphi}(a,n) \ =_{\text{def}} \ a = \ell \circ \subseteq : \\
&\qquad \{A \times \mathbb{N} \,|\, \varphi\} \xrightarrow{\subseteq} A \times \mathbb{N} \xrightarrow{\ell} A, \text{ and} \\
&\widehat{\mu}\varphi = \widehat{\mu}\varphi(a,n) \ =_{\text{def}} \ \min\{m \leq n \,|\, \varphi(a,m)\} : \\
&\qquad \{A \times \mathbb{N} \,|\, \varphi\} \subseteq A \times \mathbb{N} \to \mathbb{N}.
\end{aligned}$$

Comment: This **definition** of $\mu\varphi : A \rightharpoonup \mathbb{N}$ is a *static* one, by enumeration $(\ell, \widehat{\mu}\varphi) : \{A \times \mathbb{N} \,|\, \varphi\} \to A \times \mathbb{N}$ of its *graph,* as is the case in general here for *partial* PR maps: We start with *given* pairs in enumeration Domain $\{A \times \mathbb{N} \,|\, \varphi\}$, and get *defined arguments* a "only" as $d_{\mu\varphi}$*-enumerated* "elements" (*dependent variable*) $a = d_{\mu\varphi}(\widehat{(a,n)}) = d_{\mu\varphi}(a,n)$, $\widehat{(a,n)} = (a,n)$ "already known" to lie in $D_{\mu\varphi} = \{A \times \mathbb{N} \,|\, \varphi\}$: No need—and in general no "direct" possibility—to *decide,* for a given $a \in A$, **if** a is of form $a = d_{\mu\varphi}(a,n)$ with $(a,n) \in D_{\mu\varphi}$, i. e.if *Exists* $n \in \mathbb{N}$ such that $\varphi(a,n)$. In particular, if $D_{\mu\varphi} = \{A \times \mathbb{N} \,|\, \varphi\} = \emptyset_{A \times \mathbb{N}}$, then $d_{\mu\varphi}$ as well as $\widehat{\mu}\varphi$ are empty maps.

μ-Lemma: $\widehat{\mathbf{S}}$ admits the following (Free-Variables) schema (μ) combined with ($\mu!$)—*Uniqueness*—as a **Characterisation** of the μ-operator $\langle \varphi : A \times \mathbb{N} \to \mathbb{2} \rangle \mapsto \langle \mu\varphi : A \rightharpoonup \mathbb{N} \rangle$ above:

$$(\mu)\quad \frac{\varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbb{2}\ \mathbf{S}-\text{map (“predicate”)},}{\begin{array}{l} \mu\varphi = \langle (d_{\mu\varphi}\,,\, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N} \\ \qquad \text{is an } \widehat{\mathbf{S}}\text{-map such that} \\ \mathbf{S} \vdash\ \varphi(d_{\mu\varphi}(\hat{a}), \widehat{\mu}\varphi(\hat{a})) = \mathrm{true}_{D_{\mu\varphi}} : D_{\mu\varphi} \to \mathbb{2}, \\ +\ \text{“argumentwise” } \textbf{minimality:} \\ \mathbf{S} \vdash\ [\,\varphi(d_{\mu\varphi}(\hat{a}), n) \implies \widehat{\mu}\varphi(\hat{a}) \le n\,] : D_{\mu\varphi} \times \mathbb{N} \to \mathbb{2} \end{array}}$$

as well as **Uniqueness**—by *maximal extension*:

$$(\mu!)\quad \frac{\begin{array}{l} f = f(a) : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f} : D_f \to \mathbb{2}, \\ \mathbf{S} \vdash\ \varphi(d_f(\hat{a}), n) \implies \widehat{f}(\hat{a}) \le n : D_f \times \mathbb{N} \to \mathbb{2} \end{array}}{\mathbf{S} \vdash\ f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N} \ \ \text{(inclusion of graphs)}}$$

[Requiring this maximality of $\mu\varphi$ is *necessary*, since—for example—(μ) alone is fulfilled already by the *empty* partial function $\emptyset : A \rightharpoonup \mathbb{N}$]

Proof of $\mu\varphi : A \rightharpoonup \mathbb{N}$ to satisfy upper, “existence” part “(μ)” of the schema is straigthforward by **definition** of $\mu\varphi$. What remains to be **proved** is **Uniqueness-by-Maximal-Extension** schema $(\mu!)$:

Let a partial map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

be given such that f fullfills the antecedent of schema $(\mu!)$. Then the **PR**-map

$$j = j(\hat a) \ =_{\text{def}} \ (d_f(\hat a), \widehat{f}(\hat a)) : D_f \to A \times \mathbb{N}$$

defines in fact, by the first premise on f, namely

$$\varphi(d_f(\hat a), \widehat{f}(\hat a)) = \mathrm{true}_{D_f}(\hat a) : D_f \to 2,$$

an **S**-map $j : D_f \to \{A \times \mathbb{N} \mid \varphi\}$ which establishes the wanted graph inclusion, namely

$$j : [\, f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}\,],$$

as shows the following $\mathbf{S}/\widehat{\mathbf{S}}$-DIAGRAM:

$$\begin{tikzcd}
 & A \times \mathbb{N} \arrow[dl, "d_{\mu\varphi} = \ell"'] \arrow[d, "\varphi"] \arrow[dr, "\widehat{\mu}\varphi"] & \\
A \arrow[rr, bend right, "\mu\varphi"] \arrow[rr, bend right=40, "f"'] & 2 & \mathbb{N} \\
D_f \arrow[u, "d_f"] \arrow[uur, "j = (d_f, \widehat{f})"'] \arrow[urr, bend right, "\widehat{f}"'] & &
\end{tikzcd}$$

μ-applied-to-**S**-predicates DIAGRAM

Here, by **definition** of $\widehat{\mu}\varphi = \widehat{\mu}\varphi(a, n) : D_{\mu\varphi} = A \times \mathbb{N} \to \mathbb{N}$ above, we have in particular

$$\begin{aligned}
&\widehat{\mu}\varphi \circ j(\hat a) \ =_{\text{by def}} \ \widehat{\mu}\varphi(d_f(\hat a), \widehat{f}(\hat a)) \\
&\ = \min\{m \le d_f(\hat a) \,|\, \varphi(d_f(\hat a), m)\} : D_f \to A \times \mathbb{N} \to \mathbb{N}, \\
&\ = \widehat{f}(\hat a) : D_f \to \mathbb{N},
\end{aligned}$$

the latter by assumed *minimum property* of

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}.$$

Together with (trivial)

$$d_{\mu\varphi} \circ j \ =_{\text{by def}} \ \ell_{A,\mathbb{N}} \circ (d_f, \widehat{f}) = d_f : D_f \to A \times \mathbb{N} \to A$$

this gives in fact (remaining) *graph-inclusion* $f \,\widehat{\subseteq}\, \mu\varphi : A \rightharpoonup \mathbb{N}$ via $j = (d_f, \widehat{f}) : D_f \to D_{\mu\varphi} = A \times \mathbb{N}$ **q.e.d.**

Remark: Within PEANO-Arithmétique **PA**, and hence also within set theory, our $\mu\varphi : A \rightharpoonup \mathbb{N}$ equals

$$\mu\varphi = \langle (\subseteq, \widehat{\mu}\varphi) : \hat{A} \to A \times \mathbb{N} \rangle : A \supset \hat{A} \to \mathbb{N},$$

with $\hat{A} = \{\hat{a} \in A \,|\, \exists n \ \varphi(\hat{a}, n)\}$, and $\widehat{\mu}\varphi(\hat{a}) = \min\{m \in \mathbb{N} \,|\, \varphi(\hat{a}, m)\} : \hat{A} \to \mathbb{N}$, i. e.it is given there by the classical—partial—minimum definition. But this definition lacks *constructivity,* since $\hat{A} \subseteq A$ is in general not PR decidable.

What about the *Converse Direction* to μ-**Lemma** above? In fact:

Partial PR $\equiv$ μ-Recursion, Instance of Church's Thesis: Any *partial* **S**-map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$$

is *represented*—within theory $\widehat{\mathbf{S}}$—by an " $\widehat{=}$ " equal μ-**recursive** $\widehat{\mathbf{S}}$ map, namely by

$$g = (\widehat{f} \circ \mathrm{count}_{D_f}) \,\widehat{\circ}\, \mu\varphi_f : A \rightharpoonup \mathbb{N} \to D_f \to B,$$

$\varphi_f = \varphi_f(a, n) : A \times \mathbb{N} \to \mathbb{2}$ suitable, namely

$$\varphi_f = \varphi_f(a, n) \ =_{\text{def}} \ [\, a \ \dot{=}_A \ d_f \circ \text{count}_{D_f}(n) \,] : A \times \mathbb{N} \to \mathbb{2} \text{ (PR)},$$

$\text{count}_{D_f} : \mathbb{N} \to D_f$ being a CANTOR type (PR) *count* of D_f.

Joker Remark:

$$\text{count}_{D_f} = \text{count}_{D_f}(n) : \mathbb{N} \to D_f = \{\mathbb{X} \,|\, D_f : \mathbb{X} \to \mathbb{2}\}$$

is easily constructed if D_f comes with a *point,* $\hat{a}_0 : \mathbb{1} \to D_f$ say. If not—or if you cannot name such point—, just add one "as a joker", namely injection $\iota : \mathbb{1} \to \mathbb{1} + D_f$ into the sum, replace D_f by $\mathbb{1} + D_f$, A by $\mathbb{1} + A$, B by $\mathbb{1} + B$, d_f by $\mathbb{1} + d_f : \mathbb{1} + D_f \to \mathbb{1} + A$, $\widehat{f}$ by $\mathbb{1} + \widehat{f} : \mathbb{1} + D_f \to \mathbb{1} + B$, and keep track of the joker.

So D_f is "now" pointed, and admits—because of this—a retractive count $\text{count}_{D_f} : \mathbb{N} \to D_f$, by linear (well) order on D_f, inherited from that of $\mathbb{X}$, and anchored at D_f's *point,* "defined element" $\hat{a}_0 : \mathbb{1} \to D_f \sqsubset \mathbb{X}$.

Proof of Partials to be μ-recursive maps:

Consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ DIAGRAM:

$$\begin{tikzcd}
& D_f \arrow[d, "d_f"] \arrow[rrd, "\widehat{f}"] \arrow[ldd, bend right, "i"'] \arrow[rd, phantom, "\widehat{=}"] & & \\
\arrow[r, phantom, "="] & A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[rd, "\mu\varphi_f"] \arrow[d, phantom, "\widehat{=}"] & & B \arrow[ld, phantom, "\underset{(\text{def})}{\widehat{=}}"] \\
D_{\mu\varphi_f} \arrow[ru, "d_{\mu\varphi_f}"] \arrow[ruu, bend left, "j"] \arrow[rr, "\widehat{\mu}\varphi_f"'] & & \mathbb{N} \arrow[r, "\text{count}"'] & D_f \arrow[u, "\widehat{f}"']
\end{tikzcd}$$

Partial PR Map $\equiv$ μ-Recursion DIAGRAM

All Objects and (partial) maps in this DIAGRAM have been defined above, with the exception of (PR) *comparison maps* $i : D_f \to D_{\mu_f}$, and j in the other direction.

We define these two maps "suitably", by

$$\begin{aligned}
&D_{\mu\varphi_f} \ =_{\text{by def}} \ \{A \times \mathbb{N} \,|\, \varphi_f\} \ =_{\text{by def}} \ \{(a,n) \,|\, d_f \circ \mathrm{count}_{D_f}(n) \ \dot{=}_A \ a\},\\
&i = i(\hat{a}) \ =_{\text{def}} \ (d_f(\hat{a}), \min\{m \le n \,|\, d_f(\mathrm{count}_{D_f}) \ \dot{=}_A \ d_f(\hat{a})\} : D_f \to D_{\mu\varphi_f},\\
&\text{and } j = j(a,n) \ =_{\text{def}} \ \mathrm{count}_{D_f}(\min\{m \le n \,|\, d_f(\mathrm{count}(m)) \doteq a\}) :\\
&A \times \mathbb{N} \supseteq D_{\mu\varphi_f} \to D_f.
\end{aligned}$$

By **definition** of $\varphi_f : A \times \mathbb{N} \to 2$, and then—general for such a predicate, see above—of

$$\mu\varphi_f = \langle (d_{\mu\varphi_f}, \widehat{\mu}\varphi_f) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N},$$

and—eventually—(alledged) *representant*

$$g \ =_{\text{def}} \ \widehat{f} \circ \mathrm{count}_{D_f} \,\widehat{\circ}\, \mu\varphi_f : A \rightharpoonup \mathbb{N} \to D_f \to B,$$

of f, this $\widehat{\mathbf{S}}$-DIAGRAM commutes; μ-recursive *representant* involves just (two) **S**-maps, namely—PR retraction count $=$ $\mathrm{count}_{D_f} : \mathbb{N} \twoheadrightarrow D_f$, and *rule* $\widehat{f} : D_f \to B$ (given)—, as well as one genuinely μ-recursive map $\mu\varphi_f : A \rightharpoonup \mathbb{N}$: μ-recursion applied to **S**-predicate $\varphi_f : A \times \mathbb{N} \to 2$. Commutativity of this $\widehat{\mathbf{S}}$-DIAGRAM shows

$$i : [\, f \,\widehat{\subseteq}\, g : A \rightharpoonup B \,], \ j : [\, g \,\widehat{\subseteq}\, f : A \rightharpoonup B \,], \text{ and hence } f \,\widehat{=}\, g : A \rightharpoonup B :$$

An arbitrary *partial* PR map $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$ admits, within $\widehat{\mathbf{S}}$, a **representation** $g : A \rightharpoonup B$, obtained via suitable **S**-map(s) and one μ-recursive one, $\mu\varphi_f : A \rightharpoonup \mathbb{N}$, **defined** in turn "over" an **S**-predicate,namely $\varphi_f : A \times \mathbb{N} \to 2$ above **q.e.d.**

Corollary: Define theory $\mu\mathbf{S}$, over $\mathbf{S}$ and within $\widehat{\mathbf{S}}$, by **Closure** of $\mathbf{S}$ under the μ-operator—applied to $\mathbf{S}$-predicates—*merged* with **Monoidal-Theory Closure.** Then this *subtheory* $\mu\mathbf{S}$ is in fact isomorphic to theory $\widehat{\mathbf{S}}$, as a Diagonal Monoidal PR Theory:

$$\mathbf{S} \sqsubset \mu\mathbf{S} \cong \widehat{\mathbf{S}}.$$

Both theories have Cartesian PR theory **S embedded** as diagonal monoidal **subcategory,** and the embedding is compatible with the isomorphism $\mu\mathbf{S} \cong \widehat{\mathbf{S}}$.

By foregoing **Theorem,** and the **Closure Theorem** of chapter above for theorie(s) $\widehat{\mathbf{S}}$—strengthening Theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$— we are sure that we can extend the μ-operator—already within theory $\widehat{\mathbf{S}}$— to *partial* predicates

$$\varphi = \langle (d_\varphi, \widehat{\varphi}) : D_\varphi \to (A \times \mathbb{N}) \times \mathbb{2} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2},$$

and that the $\widehat{\mathbf{S}}$-morphism

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

—if suitably defined along Partial-Map **Closure** Terminology—inherits suitably **generalised** charateristic properties from the μ-operator applied to PR predicate φ above, i. e.to $\varphi : A \times \mathbb{N} \to \mathbb{2}$ in $\mathbf{S}$.

But may be a *direct* **definition** and **characteristic schema** for the μ-operator on (partially defined) predicates

$$\varphi = \langle \gamma\varphi = (\ell \circ \gamma\varphi, r \circ \gamma\varphi) =: (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_\varphi \to A \times \mathbb{N} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2}$$

is less complicate—and more instructive:

Definition: Given an $\widehat{\mathbf{S}}$-predicate $\varphi : A \times \mathbb{N} \rightharpoonup 2$, we take as *Enumeration Domain* $D_{\mu\varphi}$ for (the graph of) $\mu\varphi : A \rightharpoonup \mathbb{N}$ to be constructed,

$$D_{\mu\varphi} \ =_{\mathrm{def}} \ \{D_\varphi \,|\, \widehat{\varphi}\} \ =_{\mathrm{by\,def}} \ \{\alpha \in D_\varphi \,|\, \widehat{\varphi}(\alpha)\},$$

and as *components* of $\mu\varphi$:

$$\begin{aligned}
&d_{\mu\varphi} \ =_{\mathrm{def}} \ \ell \circ d_\varphi \circ \subseteq : \{D_\varphi \,|\, \widehat{\varphi}\} \to D_\varphi \to A \times \mathbb{N} \to A, \ \text{and} \\
&\widehat{\mu\varphi} = \widehat{\mu\varphi}(\alpha) \ =_{\mathrm{def}} \ \min\{m \le r \circ d_\varphi \circ \subseteq \ (\alpha)\} : \\
&\{D_\varphi \,|\, \widehat{\varphi}\} \xrightarrow{\subseteq} D_\varphi \xrightarrow{d_\varphi} A \times \mathbb{N} \to \mathbb{N} \to \mathbb{N},
\end{aligned}$$

cf. "μ-applied-to-partial-predicates DIAGRAM in **Proof** of next Theorem.

[It is obvious that **restriction** of the above *extended* μ-operator to **S**-predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \to 2$ coincides with the one given at begin of chapter for this case: In that PR case for φ, $D_\varphi = A \times \mathbb{N}$ etc.]

We now **generalise** the characteristic μ-schemata for the PR, **S**-predicates $\varphi : A \times \mathbb{N} \to 2$, to the case of *partial* ones $\varphi : A \times \mathbb{N} \rightharpoonup 2$, and **prove** that $\mu\varphi : A \rightharpoonup A$ **defined** above for that general case of φ, *fullfills* the schemata, in particular the schema of *uniqueness by maximality,* within theory $\widehat{\mathbf{S}}$ of *partial* **S**-maps.

Characterisation Theorem for the μ-operator applied to ***partial*** **predicates** $\varphi : A \times \mathbb{N} \rightharpoonup 2$:

Theory $\widehat{\mathbf{S}}$ admits characterisation of its μ-operator introduced above, by the following (general) μ-**schema:**

$$(\mu)_{\widehat{\mathbf{S}}} \quad \frac{\varphi = \langle (d_\varphi, \widehat{\varphi}) \rangle : A \times \mathbb{N} \rightharpoonup 2 \ \ \widehat{\mathbf{S}}\text{-}\textit{predicate},}{\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}}$$

an $\widehat{\mathbf{S}}$-morphism, i. e.a *partial* $\mathbf{S}$-map, such that

$$\mathrm{true}_A \mathrel{\widehat{\supseteq}} \varphi \mathbin{\widehat{\circ}} (\mathrm{id}_A \times \mu\varphi) \mathbin{\widehat{\circ}} \Delta_A : A \to A \times A \rightharpoonup (A \times \mathbb{N}) \rightharpoonup 2,$$

in diagonal monoidal Free-Variables “Calculus”:

$$\mathrm{true}_A \mathrel{\widehat{\supseteq}} \varphi \mathbin{\widehat{\circ}} \ (a, \mu\varphi \mathbin{\widehat{\circ}} a) : A \rightharpoonup 2.$$

$+$ *minimality* (FV):

$$\mathrm{true}_{A\times\mathbb{N}} \mathrel{\widehat{\supset}} [\, \varphi \mathbin{\widehat{\circ}} (a, n) \implies \mu\varphi \mathbin{\widehat{\circ}} a \le n \,] : A \times \mathbb{N} \rightharpoonup 2,$$

with *Free Variables* $a \in A$ and $n \in \mathbb{N}$ interpreted as *identities:*

$$\mathrm{true}_{A\times\mathbb{N}} \mathrel{\widehat{\supset}} \ \implies \widehat{\circ}\,(\varphi, \ \le \widehat{\circ}\,(\mu\varphi, r)) :$$
$$A \times \mathbb{N} \rightharpoonup (A \times \mathbb{N}) \times \mathbb{N}^2 \rightharpoonup 2^2 \to 2,$$

where—as usual—“induced” partial maps are just taken as **abbreviations** for the “official” versions **defined** via *diagonals.*

+ *uniqueness* of $\mu\varphi : A \rightharpoonup \mathbb{N}$ by **maximal-graph** property:

$$
(\mu!)_{\widehat{\mathbf{S}}} \quad \frac{\begin{array}{l} f : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ (given) such that} \\ f : A \rightharpoonup \mathbb{N} \text{ in place of } \varphi \text{ satisfies} \\ \text{all of the above graph inclusions,} \\ \text{(in particular } \mathit{minimality}) \end{array}}{\mathbf{S} \vdash \ f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}.}
$$

Proof: That our $\mu\varphi : A \rightharpoonup \mathbb{N}$ **defined** above for partial predicate $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$ satisfies the *basic* schemata (μ) and *minimality* above, is seen straightforward.

But what about *graph-maximality* with respect to other partial maps

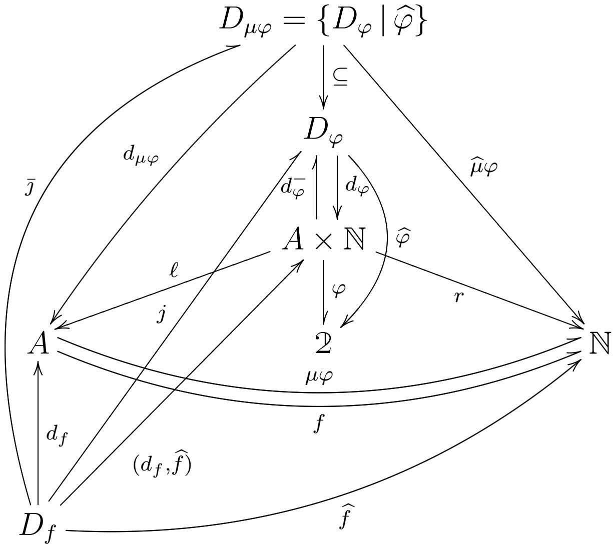
$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N},$$

equally satisfying these two conditions?

We generalise our earlier **proof** of this graph-maximality, with respect to **S**-predicates φ, to the case of a *partial* one,

$$\varphi = \langle (d_\varphi, \widehat{\mu}\varphi) : D_\varphi \to (A \times \mathbb{N}) \times \mathbb{2} \rangle : A \times \mathbb{N} \rightharpoonup \mathbb{2}, \ \text{in } \widehat{\mathbf{S}},$$

by enriching the earlier "μ-applied-to-**S** DIAGRAM" with the new data for the φ-*partial* case:



μ-applied-to-partial-predicates DIAGRAM

The lower part of the DIAGRAM is to display the data (components) of such a “candidate” $f : A \rightharpoonup \mathbb{N}$. What we need for the asserted inclusion $f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}$ is a *commutative fill-in* into the diagram, from Domain D_f to Domain $D_{\mu\varphi} = \{D_\varphi \,|\, \widehat{\varphi}\}$. By the **Closure Theorem** of foregoing chapter, a *partial map* commutative fill-in $\bar{\jmath} : D_f \rightharpoonup \{D_\varphi \,|\, \widehat{\varphi}\}$ is enough. For “constructing” this, take as a—nontrivial, new—building block, partial map $d_\varphi^- : A \times \mathbb{N} \rightharpoonup D_\varphi$, opposite (as graph) to $d_\varphi : D_\varphi \to A \times \mathbb{N}$ given, cf.**Structure Theorem** for theory $\widehat{\mathbf{S}}$, assertion (ii): Basic Partial Map DIAGRAM for $f : A \rightharpoonup B$, here for $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$:

This opposite d_φ^- has, by that **Theorem,** the *typical property*

$$\widehat{\varphi} \mathbin{\widehat{\circ}} d_\varphi^- \mathrel{\widehat{=}} \varphi : A \times \mathbb{N} \rightharpoonup D_\varphi \to \mathbb{2}.$$

So we get a precursor for realising *graph-inclusion,* namely

$$j \;=_{\mathrm{def}}\; d_\varphi^- \,\widehat{\circ}\, (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup D_\varphi.$$

Because of

$$\begin{aligned}
&\widehat{\varphi} \,\widehat{\circ}\, j : D_f \rightharpoonup D_\varphi \\
&\quad =_{\mathrm{by\,def}}\; \widehat{\varphi} \,\widehat{\circ}\, d_\varphi^- \circ (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup D_\varphi \to \mathbb{2} \\
&\quad \mathrel{\widehat{=}} \varphi \,\widehat{\circ}\, (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rightharpoonup \mathbb{2} \;\; \text{(see just above)} \\
&\quad \mathrel{\widehat{=}} \mathrm{true}_{D_f} : D_f \rightharpoonup \mathbb{2} \;\; \text{(by comparison condition on } f : A \rightharpoonup \mathbb{N}\text{)} \\
&\quad \mathrel{\widehat{\subseteq}} \mathrm{true}_A : A \to \mathbb{2} \text{ via } d_f \to A,
\end{aligned}$$

we get in particular

$$\widehat{\varphi} \,\widehat{\circ}\, j \mathrel{\widehat{=}} \mathrm{true}_{D_\varphi} : D_f \to D_\varphi \to \mathbb{2}.$$

Since the *universal* equaliser property of predicative extension

$$\{D_\varphi \,|\, \widehat{\varphi}\} \xrightarrow{\subseteq} D_\varphi,$$

equaliser of $\widehat{\varphi},\ \mathrm{true}_{D_\varphi} : D_\varphi \to \mathbb{2}$, is preserved by **embedding** $\mathbf{S} \sqsubset \widehat{\mathbf{S}}$, the above eventually "generates" the lacking $\widehat{\mathbf{S}}$ morphism

$$\bar{\jmath} : D_f \rightharpoonup D_{\mu\varphi} = \{D_\varphi \,|\, \widehat{\varphi}\}$$

which establishes, by the shown $\widehat{=}$ commutativities, (graph) *inclusion*

$$f \mathrel{\widehat{\widehat{\subseteq}}} \mu\varphi : A \rightharpoonup \mathbb{N},$$

and hence by Closure property of embedding $\sqsubseteq : \widehat{\mathbf{S}} \xrightarrow{\cong} \widehat{\widehat{\mathbf{S}}}$ the characteristic properties of the μ-operator, here in case of application to *partial* predicates $\varphi : A \times \mathbb{N} \rightharpoonup \mathbb{2}$ **q.e.d.**

[It is obvious that **Definition** and **general schema** $(\mu)_{\widehat{\mathbf{S}}}$ above, **restrict** to earlier (definition resp.) schema $(\mu) = (\mu)_{\mathbf{S}}$ for PR, **S**-predicates $\varphi : A \times \mathbb{N} \to 2$]

Our **Conclusion** so far is:

- We can *eliminate formal existential quantification* —as well as (individual, formal) *variables*— from the theory of μ-recursion, by interpreting the μ-operator from theories **S** of Primitivive Recursion into their respective extensions $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$ by *partial* PR maps.

- The μ-operator canonically extends to all *partial* predicates $\varphi : A \times \mathbb{N} \rightharpoonup 2$, and **associates** to them just partial maps $\mu\varphi : A \rightharpoonup \mathbb{N}$, within $\widehat{\mathbf{S}}$ itself. So, "once again", we see, that theories $\widehat{\mathbf{S}}$ of *partial PR maps* are *closed,* this time under the μ-operator, "in parallel" to *Closure* of $\widehat{\mathbf{S}}$ under forming *partial* maps: *partial partial PR maps* "are" *partial* PR maps.

- We have the following chain of <u>isomorphisms</u> of categorical theories considered so far:

  $$\mathbf{S} \sqsubset \mu\mathbf{S} \cong \mu\mu\mathbf{S} \cong \widehat{\widehat{\mathbf{S}}} \cong \widehat{\mathbf{S}} \sqsupset \mathbf{S},$$

  the <u>embeddings</u> being *diagonal-monoidal PR compatible* with the <u>isomorphisms</u>.

  [A *partial* PR map $f : A \rightharpoonup B$ which is, "by hazard", a **total** map—discussion of overall *termination* = *total definedness* in **chapter 2**—is in general *not* itself PR: only its graph $(d_f, f) : D_f \to A \times B$ is PR. Ackermann type maps, in particular *evaluation* of all PR-map-codes, are *counter examples.*]

- Conversely, the μ-operator, already when applied only to **S**-predicates: PR predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, *generates* all $\widehat{\mathbf{S}}$-morphisms—*partial* **S**-maps—out of **S**, via necessarily formally *partial* composition with suitable **S**-maps.

- As important special cases of basic PR theories **S** we have at the moment Theory $\mathbf{PRa} = \mathbf{PR} + \text{(abstr)}$, Universe PR Theory $\mathbf{PRa}\mathbb{X}$, as well as the PR *trace* $\mathbf{PA} \restriction \mathrm{PR}$ of **PA** : All PR maps with all those equations in between, which are derivable by **PA** : Our theories, notions, and results have a structure-preserving <u>Interpretation</u> "into" (within) Peano-Arithmetic **PA**, a fortiori into classical **set** theory.

2.5 Content Driven Loops

By a *content driven* loop we mean an *iteration* of a given *step endo map,* whose number of performed steps is not known at *entry time* into the *loop*—as is the case for a PR iteration $f^{\S}(a, n) : A \times \mathbb{N} \to A$ with *iteration number* $n \in \mathbb{N}$—, but whose (re) entry into a "new" endo step $f : A \to A$ depends on *content* $a \in A$ reached so far:

This *(re) entry* or *exit* from the loop is now *controlled* by a *(control) predicate* $\chi = \chi(a) : A \to \mathbb{2}$.

First example: a <u>while</u> loop $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$, for given PR *control* predicate $\chi = \chi(a) : A \to \mathbb{2}$, and *(looping) step* endo $f : A \to A$, both in **S**, both **S**-maps for the time being, **S** as always in our present context an extension of **PRa**, admitting the schema of (predicate) *abstraction.* Examples for the moment: $\mathbf{PRa} = \mathbf{PR} +$ (abstr) itself, Universe Theory $\mathbf{PRa}\mathbb{X}$ as well as $\mathbf{PA} \restriction \mathrm{PR}$, restriction of **PA** to its PR terms, with inheritance of all **PA**-equations for this term-restriction.

Classically, *with* variables, such $\mathrm{wh} = \mathrm{wh}\,[\,\chi \,|\, f\,]$ would be "defined"—in *pseudocode*—by

$$\begin{aligned}\mathrm{wh}(a) :=\, [\, & a' := a; \\ & \underline{\text{while}}\ \chi(a')\ \underline{\text{do}}\ a' := f(a')\ \underline{\text{od}}; \\ & \mathrm{wh}(a) := a'\,].\end{aligned}$$

The formal version of this—within a *classical,* element based setting—, is the following partial-(Peano)-map characterisation:

$$\mathrm{wh}(a) = \mathrm{wh}\,[\,\chi \,|\, f\,]\,(a) = \begin{cases} a \text{ if } \neg\,\chi(a) \\ \mathrm{wh}(f(a)) \text{ if } \chi(a) \end{cases} \quad : A \rightharpoonup A.$$

But can this *dynamical, bottom up* "definition" be converted into a PR *enumeration* of a suitable *graph* "of all *argument-value pairs*" in terms of an $\widehat{\mathbf{S}}$-morphism

$$\mathrm{wh} = \mathrm{wh}\,[\,\chi \,|\, f\,] = \langle (d_{\mathrm{wh}}, \widehat{\mathrm{wh}}) : D_{\mathrm{wh}} \to A \times A \rangle : A \rightharpoonup A?$$

In fact, we can give such *suitable,* static **Definition** of $\mathrm{wh} = \mathrm{wh}\,[\,\chi \,|\, f\,] : A \rightharpoonup A$—within $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—as follows:

$$\begin{aligned}&\mathrm{wh} \ =_{\mathrm{def}}\ f^{\S}\,\widehat{\circ}\,(\mathrm{id}_A, \mu\,\varphi_{[\,\chi\,|\,f\,]}) \\ &\quad =_{\text{by def}}\ f^{\S}\,\widehat{\circ}\,(A \times \mu\,\varphi_{[\,\chi\,|\,f\,]})\,\widehat{\circ}\,\Delta_A : \\ &A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \ \text{where} \\ &\varphi = \varphi_{[\,\chi\,|\,f\,]}(a, n) \ =_{\mathrm{def}}\ \neg\,\chi\, f^{\S}(a, n) : A \times \mathbb{N} \to A \to \mathbb{2} \to \mathbb{2}.\end{aligned}$$

Within a quantified Arithmetical Theory like **PA**, this $\widehat{\mathbf{S}}$**-Definition** of $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$ fullfills the classical **characterisation** quoted above, as is readily shown by Peano-Induction "on" $n := \mu\,\varphi_{[\,\chi\,|\,f\,]}\,(a) : A \rightharpoonup \mathbb{N}$, at least within **PA** and its extensions.

[Classically, *partial definedness* of this—*dependent*—induction parameter n causes no problem: use a *case distinction* on definedness of $\mu\,\varphi_{\chi,f}(a)$"$\in$"$\mathbb{N}$. Even in our quantifier-free context such *dependent induction* on a *partial* dependent induction parameter will be available, see below]

In this generalised sense, we have—within theories $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—all while loops, for the time being at least those with *control* $\chi : A \to \mathbb{2}$ and *step* endo $f : A \to A$ within **S**.

It is obvious that such $\mathrm{wh}\,[\,\chi\,|\,f\,] : A \times A$ is in general "only" *partial*—as is trivially exemplified by integer division by *divisor* 0, which would be endlessly subtracted from the dividend, although in this case *control* and *step* are both PR.

By the classical characterisation of these while loops above, we are motivated for its generalisation to the $\mathbf{S}/\widehat{\mathbf{S}}$ case:

Characterisation Theorem for while loops *over* **S**, within Theory $\widehat{\mathbf{S}}$: For $\chi : A \to \mathbb{2}$ (*control*) and $f : A \to A$ (*step*), both—for the time being—**S**-maps, while loop $\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$ (as **defined** above), is **characterised** by the following *implications* within $\widehat{\mathbf{S}}$:

$$\widehat{\mathbf{S}} \vdash \neg\,\chi \circ a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq a : A \rightharpoonup \mathbb{2}, \text{ and}$$
$$\widehat{\mathbf{S}} \vdash \chi \circ a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq \mathrm{wh}\,\widehat{\circ}\,f \circ a.$$

where use of "sort of" free variable 'a' is to help intuition, *formally* a is just another name for $\mathrm{id}_A : A \to A$, more precisely:

Rudiments of a Free-Variables Calculus "over" a **diagonal symmetric monoidal** (PR) **theory:**

– An **identity** id : $A \to A$—*not* a projection in general—can be seen as a *Free Variable,* "ranging over" those "arguments" of A, for which the *partial map* in question—to be "applied" to that "argument"—is "thought to be defined".

– For $a \in A,\ b \in B$ free, (a, b) is—again, as in the Cartesian case—<u>interpreted</u> as the identity $(a \times b) = id_{A\times B} : (A \times B) \to (A \times B)$, here for given *partial* maps $f : A \rightharpoonup A',\ g : B \rightharpoonup B'$ as in

$$
\begin{aligned}
&(f \times g) \mathbin{\widehat{\circ}} (a, b) \\
&\quad =_{\text{by def}} \ (f \times g) \mathbin{\widehat{\circ}} (\mathrm{id}_A \times \mathrm{id}_B) \\
&\quad \mathrel{\widehat{=}} ((f \mathbin{\widehat{\circ}} a) \times (g \mathbin{\widehat{\circ}} b)) : A \times B \rightharpoonup A' \times B'.
\end{aligned}
$$

– By *transposition* Θ, such (identity) Free-Variables a and b may be interchanged—Θ may in that case become *implicit.*

– A *diagonal* Δ may *double* such an identity Free-Variable, and become implicit in turn.

Problematic is in the *partial maps* case, introduction of terminal maps $\Pi : A \to \mathbb{1}$, and even more of **projections,** since in general, for $f : C \rightharpoonup A,\ g : C \rightharpoonup B$, say within $\widehat{\mathbf{S}}$, only

$$f \mathrel{\widehat{\supseteq}} \ell \mathbin{\widehat{\circ}} (f, g) \ =_{\text{by def}} \ (f \times g) \mathbin{\widehat{\circ}} \Delta_C : C \to C^2 \to A \times B \to A,$$

analogeously for g : in general we have genuine *graph inclusions,* since

$$D_{(f,g)} = D_{(f\times g)\mathbin{\widehat{\circ}}\Delta_C} \ =_{\text{by def}} \ D_f \cap D_g$$

is in general a non-trivial pullback, not isomorphic to C as in the Cartesian case.

So—for the time being—if you want to "use" (general) **projections** as **free variables,** you must take care of the *lack of naturality* of the projection family in a general diagonal monoidal setting, or—formally—try to replace Cartesian products $A \xleftarrow{\ell} A \times B \xrightarrow{r} B$ by *half-trivial* **pullbacks**

$$A \xleftarrow{\ell} A \underset{\mathbb{1}}{\times} B \xrightarrow{r} B\ ,$$

inherited from theory **S**. *Universality* of such a pullback *over* Object $\mathbb{1}$—with its arrows $\Pi : A \to \mathbb{1}$ and $\Pi : B \to \mathbb{1}$ given—"admits" only those $\widehat{\mathbf{S}}$ morphism pairs into its factors, which have *equal domains of defined arguments.*

Using these tentative rules for a Free-Variables Calculus Interpretation, the statement of our **Characterisation Theorem** above for while loops, takes the following purely morphism theoretic form in theory $\widehat{\mathbf{S}}$:

$$\begin{aligned} &\mathrm{true}_A \mathrel{\widehat{=}} \implies \widehat{\circ}\,(\neg\,\chi,\ \dot{=}_A\ \ \widehat{\circ}\,(\mathrm{wh}, \mathrm{id}_A)) : A \rightharpoonup A,\ \textbf{and} \\ &\mathrm{true}_A \mathrel{\widehat{=}} \implies \widehat{\circ}\,(\chi,\ \dot{=}_A\ \ \widehat{\circ}\,(\mathrm{wh}, \mathrm{wh}\,\widehat{\circ}\,f)) : A \rightharpoonup A. \end{aligned}$$

We begin with the **Proof** of wh to be **unique** with regard to fullfill the "while-characterisation", by Peano-Induction on $m := \mu[\,\chi \,|\, f\,] : A \rightharpoonup \mathbb{N}$, more precisely: by Peano-Induction on *dependent induction-parameter*

$$m \doteq \widehat{\mu}[\,\neg\,\chi \,|\, f\,](\hat{a}) : D_\mu \to \mathbb{N}.$$

This needs a **Dependent Induction Parameter Peano Induc-**

tion Schema for theory **S** which reads as follows:

$$
(P5^*)\quad \frac{\begin{array}{l}
c = c\,(a) : A \to \mathbb{N}, \text{ (complexity)} \\
\chi = \chi(a) : A \to \mathbb{2}, \text{ (predicate under consideration)}, \\
\qquad \text{both in } \mathbf{S}, \\
\mathbf{S} \vdash\ c\,(a) \doteq 0 \implies \chi(a) : A \to \mathbb{2}, \text{ (anchor)} \\
\mathbf{S} \vdash\ [\,c\,(a) \doteq n \implies \chi(a)\,] \\
\qquad \implies [\,c\,(a') \doteq s\,n \implies \chi(a')\,] : A^2 \times \mathbb{N} \to \mathbb{2} \text{ (step)}
\end{array}}{\mathbf{S} \vdash\ \chi(a) : A \to \mathbb{2}.}
$$

Proof by Peano-Induction—available in PR theories via Freyd's uniqueness: apply this Peano Induction to predicate

$$\varphi = \varphi(a, n) := [\,c\,(a) \doteq n \implies \chi(a)\,] : A \times \mathbb{N} \to \mathbb{2},$$

and get overall truth of this $\varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, and hence trivially $\mathbf{S} \vdash\ \chi : A \to \mathbb{2}$, by substitution of $c\,(a)$ into n.

Comment: This schema $(P5^*)$ holds true in all *diagonal monoidal* PR theories—for example in $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$—since Peano-Induction is a consequence already of Freyd's uniqueness (FR!), which is available by axiom—in case of extensions $\widehat{\mathbf{S}}$ of PR *Cartesian* theories **S** as **Theorem.**

For **proof** of uniqness of wh in this general, diagonal monoidal case it would certainly be helpful if we could build on a suitable **generalisation** of the *Cartesian* **Free-Variables Calculus,** generalisation to a—necessarily **restricted**—form, applicable to the *diagonal (symmetric) monoidal* case of PR theories. For "rudiments" of such an FV-Calculus see above.

We extend the μ-based **definition** of wh in $\widehat{\mathbf{S}}$ formally into the case of *partial* $\chi : A \rightharpoonup 2$ and $f : A \rightharpoonup A$ by use of the μ-operator, as follows:

$$\begin{aligned}
&\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,] \ =_{\mathrm{def}}\ f^{\S}\,\widehat{\circ}\,(\mathrm{id}_A, \mu\,\varphi_{[\chi,f]}) \\
&\quad =_{\text{by def}}\ f^{\S}\,\widehat{\circ}\,(A \times \mu\,\varphi_{[\chi,f]})\,\widehat{\circ}\,\Delta : \\
&A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \ \text{where} \\
&\varphi = \varphi_{[\chi,f]} \ =_{\mathrm{def}}\ \neg\,\widehat{\circ}\,\chi\,\widehat{\circ}\,f^{\S} : A \times \mathbb{N} \rightharpoonup A \rightharpoonup 2 \to 2,
\end{aligned}$$

this time—last line—iteration $f^{\S} : A \times \mathbb{N} \rightharpoonup A$ of *partial* $f : A \rightharpoonup A$ defined—and characterised—via the "zig/zag" way in the **proof** of **Structure Theorem** above.

Characterisation Theorem for <u>while</u> **Loops:**
For (partial) *control*

$$\begin{aligned}
&\chi \mathrel{\widehat{=}} \chi\,\widehat{\circ}\,a : A \rightharpoonup 2, \ \text{and } \textit{(endo) step} \\
&f \mathrel{\widehat{=}} f\,\widehat{\circ}\,a : A \rightharpoonup A,
\end{aligned}$$

("Content driven") <u>while</u> loop

$$\mathrm{wh} = \mathrm{wh}\,[\,\chi\,|\,f\,] : A \rightharpoonup A$$

is **characterised**—within theory $\widehat{\mathbf{S}}$ of partial $\mathbf{S}$ maps—by

$$\begin{aligned}
&\mathrm{true}_A \mathrel{\widehat{=}} [\,\neg\,\chi\,\widehat{\circ}\,a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq a\,] : A \rightharpoonup 2 \ \textbf{and} \\
&\mathrm{true}_A \mathrel{\widehat{=}} [\,\chi\,\widehat{\circ}\,a \implies \mathrm{wh}\,\widehat{\circ}\,a \doteq \mathrm{wh}\,\widehat{\circ}\,f\,\widehat{\circ}\,a\,] : A \rightharpoonup 2.
\end{aligned}$$

Proof: We use the following abbreviations:

$$\begin{aligned}&\mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a := \mathrm{wh}\,[\,\chi, f\,] \mathbin{\widehat{\circ}} a : A \rightharpoonup A, \text{ and}\\ &\mu \mathrel{\widehat{=}} \mu \mathbin{\widehat{\circ}} a := \mu\{A \times \mathbb{N} \,|\, \neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S}\}\\ &\ \mathrel{\widehat{=}} \mu\{(a, n) \in A \times \mathbb{N} \,|\, \neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S} \mathbin{\widehat{\circ}} (a, n)\}.\end{aligned}$$

That $\mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a : A \rightharpoonup A$ fullfills the implications of (alledged) **characterisation** is obvious.

For showing **uniqueness** of such partial map wh, **assume** given "another" partial map $h : A \rightharpoonup A$, equally satisfying these equations, namely:

$$\begin{aligned}&\mathrm{true}_A \mathrel{\widehat{=}} [\neg\, \chi \mathbin{\widehat{\circ}} a \implies h \mathbin{\widehat{\circ}} a \mathrel{\widehat{=}} a :]A \rightharpoonup \mathbb{2} \text{ (halt), and}\\ &\mathrm{true}_A \mathrel{\widehat{=}} [\,\chi \mathbin{\widehat{\circ}} a \implies h \mathbin{\widehat{\circ}} a \mathrel{\widehat{=}} h \mathbin{\widehat{\circ}} f \mathbin{\widehat{\circ}} a\,] : A \rightharpoonup \mathbb{2}. \text{ (progress)}\end{aligned}$$

What we want to show is

$$\begin{aligned}&h \mathrel{\widehat{=}} \mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \mathbin{\widehat{\circ}} a \ =_{\text{by def}}\ f^{\S} \mathbin{\widehat{\circ}} (a, \mu \mathbin{\widehat{\circ}} a)\\ &\ =_{\text{by def}}\ f^{\S} \mathbin{\widehat{\circ}} (a, \mu\,[\,\neg \mathbin{\widehat{\circ}} \chi \mathbin{\widehat{\circ}} f^{\S}\,] \mathbin{\widehat{\circ}} a) : A \rightharpoonup A.\end{aligned}$$

The **proof** of $h \mathrel{\widehat{=}} \mathrm{wh} : A \rightharpoonup A$ is by **dependent Peano Induction,** within (diagonal monoidal) PR theory $\widehat{\mathbf{S}}$ with its **uniqueness** $(\mathrm{FR!})_{\widehat{\mathbf{S}}}$ of *initialised iterated*—inherited from basic, axiomatic version $(\mathrm{FR!})_{\mathbf{S}}$ for theory **S**.

As—dependent—induction paramter we choose $m := \mu \mathbin{\widehat{\circ}} a : A \rightharpoonup \mathbb{N}$, abbreviation see above.

Dependent Peano **anchoring:**

$$m \doteq 0 \Rightarrow:\ \neg\, \chi \mathbin{\widehat{\circ}} a \ \wedge\ [\,h \mathbin{\widehat{\circ}} a \ \doteq_A\ a\,] \ \wedge\ [\,\mathrm{wh} \mathbin{\widehat{\circ}} a \ \doteq_A a\,] : A \times \mathbb{N} \rightharpoonup \mathbb{2}.$$

This **proves** the wanted *uniqueness* in the *anchor, i. e.halt* case $m \doteq 0$.

Dependent Peano **step** implication—to be shown now— is the following:

$$\begin{aligned}\mathrm{true}_{A^2\times\mathbb{N}} \mathrel{\widehat{=}} [\,\mu \,\widehat{\circ}\, a \ \doteq m \implies h \,\widehat{\circ}\, a \ \doteq_A \ \mathrm{wh} \,\widehat{\circ}\, a\,] \\ \Rightarrow [\,\mu \,\widehat{\circ}\, a' \doteq s\,m \implies h \,\widehat{\circ}\, a' \doteq_A \mathrm{wh} \,\widehat{\circ}\, a'\,] : \\ A^2 \times \mathbb{N} \rightharpoonup \mathbb{2}.\end{aligned}$$

Conclusio of this asserted **step implication** follows from its **premise** via

$$\mu \,\widehat{\circ}\, a' \ \doteq \ s\,m \Rightarrow :$$

$$\mu \ \widehat{\circ}\, f \,\widehat{\circ}\, a' \ \doteq m$$

by bottom up characterisation of PR iteration

and hence of the μ-operator

$$\wedge \ h \,\widehat{\circ}\, a' \ \doteq_A \ h \,\widehat{\circ}\, f \,\widehat{\circ}\, a' \ \doteq_A \ \mathrm{wh} \,\widehat{\circ}\, f \,\widehat{\circ}\, a' \doteq_A \mathrm{wh} \,\widehat{\circ}\, a' : A \times \mathbb{N} \rightharpoonup \mathbb{2},$$

the latter line following from

$$\begin{aligned}&\mu \,\widehat{\circ}\, a' \ \doteq \ s\,m \implies \chi \,\widehat{\circ}\, a' \\ &\implies \mathrm{wh} \,\widehat{\circ}\, a' \ \doteq_A \ \mathrm{wh} \,\widehat{\circ}\, f \,\widehat{\circ}\, a' \\ &\qquad \wedge \ h \,\widehat{\circ}\, a' \ \doteq_A \ h \,\widehat{\circ}\, f \,\widehat{\circ}\, a' : A \times \mathbb{N} \rightharpoonup \mathbb{2}.\end{aligned}$$

A similar treatment **formalises** <u>until</u> loops: **pseudocode** for such

$$\mathrm{utl} = \underline{\mathrm{do}}\ f\ \underline{\mathrm{until}}\ \chi\ \underline{\mathrm{od}} \text{ is}$$

$$\begin{aligned}\mathrm{utl}(a) := [\,&a' := f(a); \\ &\underline{\mathrm{while}}\ \neg\,\chi(a')\ \underline{\mathrm{do}}\ a' := f(a')\ \underline{\mathrm{od}}; \\ &\mathrm{utl}(a) := a'\,].\end{aligned}$$

Definition as a *partial* PR map:

$$\mathrm{utl}\,[\,f\,|\,\chi\,](a) \;=_{\mathrm{def}}\; \mathrm{wh}\,[\,\neg\,\chi\,|\,f\,]\,\widehat{\circ}\,f : A \rightharpoonup A \rightharpoonup A.$$

This is already the general case: both f and χ possibly *partial.* Specialisation to **S**-maps f and χ looks similar: first factor $f : A \to A$ in **S**, second factor in general *partial,* in $\widehat{\mathbf{S}}$, despite of a—f and χ possibly assumed both to be in **S**.

Characterisation:

$$\chi\,\widehat{\circ}\,f\,\widehat{\circ}\,a \implies \mathrm{utl}\,[\,f\,|\,\chi\,] \;\dot{=}_A\; f\,\widehat{\circ}\,a,$$
$$\neg\,\chi\,\widehat{\circ}\,f\,\widehat{\circ}\,a \implies \mathrm{utl}\,[\,f\,|\,\chi\,]\,\widehat{\circ}\,a \;\dot{=}_A\; \mathrm{utl}\,[\,f\,|\,\chi\,]\,\widehat{\circ}\,f\,\widehat{\circ}\,a.$$

Everything proven for wh above holds—mutatis mutandis—for <u>until</u> loops $\mathrm{utl}\,[\,f\,|\,\chi\,] : A \rightharpoonup A$.

With our "full" embedding of μ-recursive maps (over a theory **S**), into (categorical) theory $\widehat{\mathbf{S}}$ of partial **S**-maps, and the *converse result*—see **Proof** above, that each *partial* **S**-map—*morphism* in theory $\widehat{\mathbf{S}}$—has, within $\widehat{\mathbf{S}}$, a representation as a μ-recursive map "over" **S**, we arrive at

A Further Case of Church's Thesis:

- The *concept* of a **partial PR map** is equivalent to that of a μ-recursive (partial) map. It is another—Free-Variables, formally: variable-free—notion of a **general recursive (partial) map,** all this in (and over) the categorical framework of an (arbitrary) *Cartesian PR theory* **S** with (schema of) abstraction of its predicates—as well as with equality *predicates* on those Objects B which are common Codomain of map pairs $f, g : A \to B$

taken into consideration, such that for these equality predicates $[\,b \doteq b'\,] : B^2 \to 2$ *Equality Definability* is guaranteed, main examples: all Objects of $\mathbf{PRa} = \mathbf{PR} +$ (abstr), and of Universe PR Theory $\mathbf{PRa}\mathbb{X}$.

This statement is slightly more general than the one(s) **proven:** Explicitely, we have considered just theories $\widehat{\mathbf{S}}$ extending specific Theory $\mathbf{PRa} = \mathbf{PR}+$(abstr) of (categorical) theory of Primitive Recursion with *predicate abstraction* and *their* extension by *partial maps.* But closer analysis of **concepts** and **proofs** shows that everything works for "basic" theory $\mathbf{S}$ taken a *Cartesian* PR theory as just described.

- Same for <u>while</u> loops $\mathrm{wh} = \mathrm{wh}\,[\,\chi \,|\, f\,] : A \rightharpoonup 2$: They obviously *generate* all μ-recursive (partial) maps: For given (PR or partial PR) predicate $\varphi : A \times \mathbb{N} \rightharpoonup 2$,

$$\mu\varphi \mathrel{\widehat{=}} r \,\widehat{\circ}\, \mathrm{wh}\,[\,\neg\,\varphi\,|\,(A \times s)\,] : A \times \mathbb{N} \rightharpoonup A \times \mathbb{N} \to \mathbb{N}$$

  satisfies the characteristic implications for the μ-operator.

  Therefore the <u>while</u>-operator wh generates all *partial* maps, in $\widehat{\mathbf{S}} \sqsupset \mathbf{S}$, even in just one step out of predicate/endo pairs $\chi : A \to 2$ and $f : A \to A$ in $\mathbf{S}$.

- Theory $\widehat{\mathbf{S}}$ is **closed** under the <u>while</u>-operator, as it is—and because it is—under the μ-operator.

- Formal Consequence of the last two assertions is in particular a fact known since long time to Computer Scientists: "one <u>while</u>

loop is enough", starting from suitable for loop programs to define **S**-maps $\chi : A \to 2$ and $f : A \to A$, "data" for a while loop $\mathrm{wh}[\,\chi \mid f\,] : A \rightharpoonup A$.

Since for loops—equivalent to PR maps—can in turn be written as (trivial) while loops,while-**Closure** of the fundamental maps: 0, s, as well as substitutions—*logical functions* in the sense of Eilenberg & Elgot—reaches all of $\mu\mathbf{S}$, but presumably not in while nesting depth 1, as is the case when starting with all for loops, see above. I guess, for such a one-step closure by the while-operator, you need at least *case distinctions,* and these come in here—formally—as PR maps on their own right, namely as *induced* maps out of a *sum* $A \xrightarrow{i} A + B \xleftarrow{j} B$.

From a **logical** point of view, there are—at least—the following two open **Questions,** in

Arithmetics Complexity Problem:

- Does Theory **PR** admit *strict, consistent* strengthenings, or is it a *simple theory,* will say that it admits its given notion of equality and the indiscrete (inconsistency) equality as only "congruences?", cf.a simple *group* which has as *normal subgroups* only itself and $\{1\}$. Because of reasons to be explained in later work, my guess here is: **PR** *admits* non-trivial strengthenings, in particular I suppose that the PR *trace* of **PA**, explained above, is a strict strengthening of **PR** resp.$\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$.

  We cannot exclude at present that all of these strengthening extensions of **PR** make up a whole **lattice** of (Free-Variables)

Arithmetical Theories, each of them giving particular, "new" features to Primitive Recursive Arithmetics.

- Already at start we possibly have such a strengthening: If Free-Variables ("Free Variables" in the classical sense) *Primitive Recursive Arithmetic* **PRA** is **defined** to have as its terms all map terms obtainable by the (full) schema of Primitive Recursion, and as formulae just the *defining equations* for the maps introduced by that schema, then I see no way to **prove** all of the usual semiring equations for $\mathbb{N}$:

  We *need* Freyd's **uniqueness** (FR!) of the *initialised iterated:* From this HORN clause we can show (!) in particular GOODSTEIN's uniqueness rules U_1 to U_4 on which *his* Proof of the semiring properties of $\mathbb{N}$ is based. He takes these rules as ***axioms.***

  My guess is here—if I have understood right the definition of **PRA**—that **PR** = **PR**+(FR!) is a strict strengthening of **PRA**, at least if there is no "underground" connection to the set theoretic view of maps as (possibly infinite) *argument-value tables.*

- Finally, *Descent* Theory $\pi\mathbf{R}$ in PART C below—defined by **axiom** of *non-infinite iterative Descent*—presumably is a strict strengthening of Theory **PRa**. It is not excluded that Theory $\pi\mathbf{R}$ is *simple* in the sense above, or can at least be strengthened into a simple Theory by a stronger Descent axiom as for example iterative non-infinite Descent in Ordinal $\mathbb{U}$ of *nested strings.*

Chapter 3

Universal Sets and Universe Theories

Our *Universal Objects* will both be defined as predicative sets of (Natural Number *codes* of) bracketed strings. We first define strings of natural numbers, order them lexicographically, and interprete them as (coefficient strings of) polynomials in one indeterminate. In next chapter, on *Iterative Evaluation,* these polynomials will measure complexity of map codes, complexity descending with each evaluation *step.*

3.1 Strings as Polynomials

Strings $a_0\, a_1\, \ldots\, a_n$ of natural numbers (in set $\mathbb{N}^+ = \mathbb{N}^* \smallsetminus \{\Box\}$ of non-empty strings) are coded as *prime power products*

$$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_{>0} \subset \mathbb{N},\ p_j \text{ the } j\,\text{th prime number,}$$

and identified with (the coefficient lists of) “their” *polynomials*

$$p(X) = a_0 + a_1 X^1 + \ldots a_n X^n$$

as well as

$$p(\omega) = a_0 + a_1 \omega^1 + \ldots a_n \omega^n,$$

in *indeterminate* X resp. ω, the latter in case that the order aspect—lexicographic order of strings—prevails. Polynomials/*polynomial strings,* out of

$$\mathbb{N}[X] \equiv \mathbb{N}[\omega] \equiv \mathbb{N}^+ \equiv \mathbb{N}_{>0} \subset \mathbb{N},$$

don't have trailing zeroes, except $p(\omega) = 0 \cdot \omega^0 \equiv 2^0$.

Addition of such polynomials is coefficient-wise/exponent-wise in codes. What we will still need is multiplication—CAUCHY product of polynomials—in particular multiplication with *indeterminate* ω, the latter being effected by right shift of all exponents:

$$p(\omega) \cdot \omega = \sum_{j=0}^{n} a_j \omega^{j+1} \equiv \prod_{j=0}^{n} p_{n+1}^{a_n}.$$

Later within present chapter we proceed—again via prime power product and prime factor decomposition—to PR construction of Universal Object $\mathbb{X}$, of (codes of) nested pairs $\mathbb{X} \subset \mathbb{N}$ which is to contain, injectively embedded, any Object of **PR** as well as of **PRa**. This Universal Object is needed for an untyped theory approach in next sections, and that approach in turn is good for a one-map definition of evaluation in next chapter.

We now go into the **formal details—skip** in first reading this **section remainder** up to **Polynomials Structure Theorem** at end—of PR definition of the sets

$$\mathbb{N}^*, \mathbb{N}^+ = \mathbb{N}^* \smallsetminus \{\Box\} \subset \mathbb{N}$$

of (codes of) strings and non-empty strings, ordered lexicographically, and interpreted as predicative subset

$$\mathbb{N}[X] \equiv \mathbb{N}[\omega] \equiv \mathbb{N}^+ \subset \mathbb{N}^* \equiv \mathbb{N}$$

of (coefficient lists of) polynomials in one indeterminate X resp. ω : $\omega = \omega^1$ is later interpreted as first infinite Ordinal, "Ordinal" $\mathbb{N}[\omega]$ will host all *complexities* of PR map codes.

In particular for integrating the codes of the brackets, (and), with the (other) "elements" $n \in \mathbb{N}$—to be represented later as *numerals* $\nu(n)$—we use a *basic coding* of *strings of natural numbers* as *prime power products.*

Recall[1] the following **Definitions** within Theory **PRa** :

- Predicate prime = prime$(n) : \mathbb{N} \to \mathbb{2}$: n is a *prime,* is **defined**—PR—as

  $$\mathrm{prime}(n) \ =_{\mathrm{def}} \ [\, n > 1\,] \wedge [\, \wedge_{i=2}^{n \dot{-} 1} \neg\, [\, i|n\,]\,],$$

  where $i|n$ PR **defines** "i is a divisor of n".

  [In our approach we express finite (!) universal quantification $\forall_{i=2}^{n}$ by iterated conjunction $\wedge_{i=2}^{n}$]

[1] from Pfender & Kröplin & Pape 1994

- **Enumeration of all prime numbers,** defined PR by

$$
\begin{aligned}
& p_0 \ =_{\text{by def}} \ 2, \\
& p_{n+1} = p_{s\,n} \\
& \quad =_{\text{by def}} \ \min_{p'}\{p_n < p' \leq (\prod_{i=0}^{n} p_n - 1) \wedge \text{prime}(p')\} : \\
& \mathbb{N} \to \mathbb{P} \ =_{\text{def}} \ \{\mathbb{N} \,|\, \text{prime}\}.
\end{aligned}
$$

This enumeration has an *inverse isomorphism*

$$\text{ord} = \text{ord}(p) \ =_{\text{def}} \ \min\{n \leq p \,|\, p_n \doteq p\} : \mathbb{P} \to \mathbb{N}.$$

- Set $\mathbb{N}^*$ of ("naked"), *basic* strings u, with their *length, first and last members,* and their *tails* and *bodies* are **defined** jointly PR as follows:

Anchoring at empty string $\in \mathbb{N}^* \equiv \mathbb{N}$:

$$
\begin{aligned}
& \Box \in \mathbb{N}^*, \textit{coded} \text{ as } \Box \equiv 0, \\
& \text{length}(\Box) \ =_{\text{def}} \ 0 \in \mathbb{N}, \\
& \text{first}(\Box) = \text{last}(\Box) = \text{tail}(\Box) = \text{body}(\Box) \ =_{\text{def}} \ \bot \equiv 0 : \\
& 0 \in \mathbb{N} \text{ in the role of } \textit{undefined value.}
\end{aligned}
$$

[We expect that the roles of $0 \in \mathbb{N}$ as first: natural number $0 \in \mathbb{N}$, second: as empty string $\Box \in \mathbb{N}^*$, and third: as undefined value $\bot$ will not conflict, here and elsewhere. Later on, we will introduce—on higher coding level—a dedicated non-empty string, $\underline{\bot}$, symbolising then used undefined value]

Recursive Step, by representation of single natural numbers as one-member strings, namely by exponents for "next free" prime:

$$\Box\, v \;\;=_{\text{by def}}\;\; \Box * v \;\;=_{\text{def}}\;\; v = v\,\Box.$$

[$\Box$ is neutral for concatenation, this definition deviates from that below for concatenation of non-empty strings, out of $\mathbb{N}^+ \equiv \mathbb{N}_{>0}$]

In fact, for $u \in \mathbb{N}^+$, "NNO letter" $b \in \mathbb{N}$:

$$\begin{aligned}
&u\, b\ [\equiv a_0\, a_1\ \ldots\ a_n\, b\ ] \\
&\quad =_{\text{def}}\;\; u * \bar{b} \;\;=_{\text{def}}\;\; u \cdot p^{b}_{\text{length}(u)} \in \mathbb{N}^+, \\
&\text{length}(u\, b) = \text{length}(u * \bar{b}) \;\;=_{\text{def}}\;\; \text{length}(u) + 1.
\end{aligned}$$

Meaning of *concatenation* with one—NNO—*letter:* Take the first prime not used for coding u to the (power) $b \in \mathbb{N}$, and *multiply.*

$$\text{first}(u\, b) = \text{first}(u * \bar{b}) \;\;=_{\text{def}}\;\; \begin{cases} b = \text{if } u \doteq \Box \equiv 0, \\ \text{first}(u),\ \text{if } u \in \mathbb{N}^+ \end{cases}$$

So strings $u \in \mathbb{N}^* \equiv \mathbb{N}$ are **displayed** as

$u \equiv \Box$ for $\text{length}(u) \doteq 0$,

$u \equiv a_0 \in \mathbb{N}$ for $\text{length}(u) \doteq 1$,

and recursively: $u \equiv a_0\ \ldots\ a_n$ for $n := \text{length}(u) - 1 > 0$.

a_j is the jth member—NNO letter—of such a string u, and results from *projection* $\pi_j(u) = \pi(j, u) : \mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$, out of $u \in \mathbb{N}^+$ and $j \in \mathbb{N}$.

Exception: For $j > \mathrm{length}(n) - 1$ projection $\pi_j(u) \ =_{\mathrm{def}}\ \bot \equiv 0 \in \mathbb{N}$ is *defined undefined:* $0 \in \mathbb{N}$ is to code the *undefined value* $\bot$, here of a projection.

In particular $\bar{\mathbb{N}} \ =_{\mathrm{def}}\ \{2^a \in \mathbb{N} \,|\, a \in \mathbb{N}\} \subset \mathbb{N}^+$ is to represent *numbers* in $\mathbb{N}$ isomorphically as *strings* of length 1, in $\bar{\mathbb{N}} \subset \mathbb{N}^+ \subset \mathbb{N}^* \equiv \mathbb{N}^+ \dot{\cup} \{\Box\}$.

Furthermore:

$$\mathrm{first}(a_0\, a_1\, \ldots\, a_n) \ =_{\mathrm{def}}\ a_0 : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N};$$
$$\mathrm{last}(a_0\, a_1\, \ldots\, a_n) = a_n : \mathbb{N}^+ \to \mathbb{N},$$
$$\mathrm{tail}(u) = \mathrm{tail}(a_0\, a_1\, \ldots\, a_n) = a_1\, \ldots\, a_n : \mathbb{N}^+ \to \mathbb{N}^*,$$
$$\mathrm{body}(u) = \mathrm{body}(a_0\, a_1\, \ldots\, a_n) = a_0\, \ldots\, a_{n-1} : \mathbb{N}^+ \to \mathbb{N}^* :$$

string u with last member deleted.

Concatenation with strings of length 1 gives, recursively, general string *concatenation* $\mathrm{cat} = \mathrm{cat}(u, v) : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$, **defined**—on codes—by

$$\mathrm{cat}(u, \Box) = u * \Box \ =_{\mathrm{def}}\ u \text{ (anchor)},$$
$$\mathrm{cat}(u, \bar{b}) = \mathrm{cat}(u, 2^b) = u * 2^b \ =_{\mathrm{def}}\ u \cdot p^b_{\mathrm{length}(u)},$$
$$\mathrm{cat}(u, v * \bar{a}) \ =_{\mathrm{by\,def}}\ (u * v) * \bar{a} \ =_{\mathrm{def}}\ \mathrm{cat}(\mathrm{cat}(u, v), a);$$

$\mathrm{last}(u * v)$, $\mathrm{body}(u * v)$, $\mathrm{tail}(u * v)$, and $\mathrm{first}(u * v)$ then are *characterised* by the obvious case distinctions for $u, v \in \mathbb{N} \equiv \mathbb{N}^* = \mathbb{N}^+ \dot{\cup} \{\Box\}$.

For these operations we obtain all of the properties expected.

Polynomials: strings in $\mathbb{N}^+ \equiv \mathbb{N}_{>0}$ have an interpretation as coefficient lists of *polynomials* $p(X) \in \mathbb{N}[X]$ in one *indeterminate,* here X.

Comments:

• Forget in this context about earlier use of "p" for a *prime number.* These prime numbers appear from now on only *indexed:* $p_0,\ p_j,\ p_n$. As name for the *indeterminate* one usually takes "x" or "y" or "z", the latter in case of extension of *scalar domain* $\mathbb{N}$ to $\mathbb{Z}$ or the complex numbers $\mathbb{C}$. But this notation suggests use of indeterminates as (free) *variables,* into which concrete numbers are *substituted,* and the polynomials then evaluated on these "concrete" *arguments,* PR, for example by Horner's schema.

What we need "here", besides monomial-wise addition of polynomials, and multiplication with the indeterminate, is canonical ("well"-) order of $\mathbb{N}[X]$, written because of this in later context '$\mathbb{N}[\omega]$': In this (backwards) lexicographical order a scalar $a \in \mathbb{N}$, seen as one-letter string $\bar{a} \in \bar{\mathbb{N}} \subset \mathbb{N}^+ \equiv \mathbb{N}[\omega]$, is smaller than $\omega \equiv \omega^1 \equiv 1 \cdot \omega^1 < b \cdot \omega^2$ etc.

• Intermediate (!) zero's in *string notation* are meaningfull: here powers of (some) primes—*monomes*—are to be skipped during coding.

In *monome* notation, these zeros can be left out.

Trailing zeros are *without* meaning for the corresponding polynomials. So our **convention** is to consider only those strings as (coefficient strings of) *polynomials* which come without trailing zeros. This makes unique the description of polynomials by their coefficient strings (out of $\mathbb{N}^+$).

Formal **Definition:** The set $\mathbb{N}[X] \equiv \mathbb{N}[\omega]$ of *polynomials*—in one indeterminate, with NNO coefficients—is **defined** as

$$\mathbb{N}[X] \equiv \{0 \cdot X^0\} \cup \{q(X) \in \mathbb{N}^+ \,|\, \mathrm{pivot}(q(X)) \neq 0, \text{ i.e. } \mathrm{last}(q) \neq 0\}.$$

Let us now prepare PR **definition** of the "canonical" (lexicographical) order on $\mathbb{N}[X]$: **Define** the *degree* as *defined partial* **PR** map $\deg[\,p(X)\,] : \mathbb{N}[X] \rightharpoonup \mathbb{N}$ by

$$\deg(p(X)) = \deg(\sum_{j=0}^{n} a_j \cdot X^n) \;=_{\mathrm{def}}\; \begin{cases} \perp \text{ if } p(X) \doteq 0 \cdot X^0, \\ n\,[= \mathrm{length}(a_0 \dots a_n) - 1\,] \\ \textit{otherwise} \end{cases}$$

Defined partiality can be expressed by extending *Codomain* for map deg into

$$\mathbb{N}\dot{\cup}\{\perp\} \;=_{\mathrm{def}}\; \bar{\mathbb{N}} \cup \{\perp\} \;=_{\mathrm{by\,def}}\; \{2^a \,|\, a \in \mathbb{N}\} \cup \{0\} \subset \mathbb{N}.$$

Object $\mathbb{N}^* \equiv \mathbb{N}_{>0}$ has no need for zero $0 \in \mathbb{N}$, for its *prime power product* coding of (coefficient) strings. So we can in fact take $\mathbb{N}$'s zero as *defined* "waste" $\perp := 0 \in \mathbb{N}$ for "values" of *defined undefined* arguments of *partial* maps such as $\deg = \deg[\,p(X)\,] : \mathbb{N}[X] \rightharpoonup \mathbb{N}$ above.

Comment: On present basic level, *non-polynomial* $\perp \equiv 0 \in \mathbb{N}$, $\mathbb{N}^* \dot{\cup} \{0\} \equiv \mathbb{N}$, plays the role of "defined" waste, this for the moment in a (formal, combinatoric) theory of polynomials: We will introduce a "higher order" waste $\underline{\perp} : \mathbb{1} \to \mathbb{N}^+$ within $\mathbb{N}^+$ "itself" later: This latter one will be needed for *defined partial* endo maps of our universal Object $\mathbb{X}$, of *(bracketed), nested pairs,* as its "own" appropriate waste, to give *full Universal Object* $\mathbb{X}_{\perp}$, of NNO numerals, (nested) pairs of numerals, and (!) coming with trash $\{\perp\}$.

For things to come in the narrower sense, explicit **definition** of *degree* of polynomials could have been avoided. In particular, in view of our order purposes we could have neglected *undefinedness* of degree for the zero-polynomial. Mais on ne sais jamais: In *Algebra* this exception is necessary for a simple definition of *polynome-division:* You cannot divide by the zero polynome, and "therefore" its degree should be *defined undefined.*

By the way, this simple case of a *defined* undefined (PR) map gives us a first challenge, how to turn such a map soundly into a *totally* defined one: the receipt is suitable extension of its *Codomain* by a *naturally* available trash: In our case it comes naturally into "Hilbert's hotel", because string coding already made free $\mathbb{N}$'s zero 0.

This given, we can now easily **define**—recursively—the (natural, linear) *("well"-)order* on the polynomial Object $\mathbb{N}[X]$ of *fundamental* Theory **PR** below: Here it is important, that scalar domain $\mathbb{N}$ already carries a *natural*—linear—well order, contrarily e.g. to $\mathbb{C}$, and even to $\mathbb{Z}$, since the usual order on $\mathbb{Z}$ is still *linear,* but *not well-founded:* $\mathbb{Z}$ has infinitely descending chains.

So here is the **definition** of the (canonical) *"well" order* of $\mathbb{N}[X]$, *intuitively* a *well* order, since nobody is expected to point to a (provably) infinitely descending chain in that (linear) order:

Dominated recursive—hence PR—**definition:**

$$
\begin{aligned}
&[p(X) \equiv a_0 a_1 \dots a_m < b_0 b_1 \dots b_n \equiv q(X)] \\
&\quad \iff \begin{cases} \deg(p(X)) < \deg(q(X)) \\ \vee\, [\deg(p(X) \doteq \deg(q(X)) \;\wedge\; \mathrm{last}(p(X)) < \mathrm{last}(q(X))] \\ \vee\, [\deg(p(X) \doteq \deg(q(X)) \;\wedge\; \mathrm{last}(p(X)) \doteq \mathrm{last}(q(X))] \\ \quad \wedge\, \mathrm{body}(p(X)) < \mathrm{body}(q(X)). \end{cases}
\end{aligned}
$$

Polynomials Structure Theorem: Polynomials $\mathbb{N}[X] \equiv \mathbb{N}[\omega] \equiv \mathbb{N}^+ \equiv \mathbb{N}_{>0}$ form a linearily ordered euclidean semiring (with componentwise defined truncated subtraction), addition and multiplication are (strictly) monotonic. Intuitively, and provably in **set** theory, this order is a *Descent* order: it admits only finite descending chains.

3.2 Universal Object $\mathbb{X}$ of numerals and nested pairs

We begin the construction of Universal Object by internal *numeralisation* of all Objective natural numbers, of Objective numerals

$$
\begin{aligned}
&\mathrm{num}(0) \equiv 0 : \mathbb{1} \to \mathbb{N}, \\
&\mathrm{num}(1) \equiv 1 \;=_{\mathrm{def}}\; (s(0)) : \mathbb{1} \to \mathbb{N} \to \mathbb{N}, \\
&\mathrm{num}(2) \equiv 2 \;=_{\mathrm{def}}\; (s(s(0)) : \mathbb{1} \to \mathbb{N} \\
&\text{etc. PR.}
\end{aligned}
$$

Internal numerals, *numeralisation*

$$\nu = \nu(n) : \mathbb{N} \to \mathbb{N}^+ \equiv \mathbb{N}_{>0} \subset \mathbb{N} :$$

$$\nu(0) \ =_{\mathrm{def}} \ \ulcorner 0 \urcorner : \mathbb{1} \to \mathbb{N} \text{ code } (\textit{goedel number}) \text{ of } 0,$$
$$\nu(1) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(0) \rangle,$$

abbreviation for (string) goedelisation, here in particular for LaTeXsource code

$$\ulcorner (\urcorner \ \ulcorner s \urcorner \ulcorner \circ \urcorner \nu(0) \ulcorner) \urcorner \ = \ \ulcorner (\urcorner \ \ulcorner s \urcorner \ulcorner \circ \urcorner \ulcorner 0 \urcorner \ \ulcorner) \urcorner$$
$$[\ \equiv 2^{\mathbf{ASCII}[(]} \ 3^{\mathbf{ASCII}[s]} \ 5^{\mathbf{ASCII}[\backslash \mathrm{circ}]} \ 7^{\mathbf{ASCII}[0]} \ 11^{\mathbf{ASCII}[)]} \in \mathbb{N}^{+} \ ],$$

$$\nu(2) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(1) \rangle = \langle \ulcorner s \urcorner \odot \langle \ulcorner s \urcorner \odot \nu(0) \rangle \rangle \quad \text{etc. PR:}$$
$$\nu(n+1) \ =_{\mathrm{def}} \ \langle \ulcorner s \urcorner \odot \nu(n) \rangle \in \mathbb{N}^{+}.$$

$\nu(n)$ has n closing brackets (at end).

This internal numeralisation distributes the "elements", numbers of the NNO $\mathbb{N}$, with suitable gaps over $\mathbb{N}$: the gaps then will receive in particular codes of any other symbols of Object Languages **PR** and **PRa** as well as of Universe Languages **PR**$\mathbb{X}$ and **PRa**$\mathbb{X}$ to come.

Object $\dot{\mathbb{N}} = \nu\mathbb{N} \subset \mathbb{N}^{+}$ of internal numerals:

Internal *numeralisation* $\nu = \nu(n) : \mathbb{N} \to \mathbb{N}$ has an *image predicate* $\chi\nu = \chi\nu(c) : \mathbb{N} \to \mathbb{2}$ —a *characteristic map* in terms of Topos Theory—PR **defined** by

$$\chi\nu(c) \ =_{\mathrm{def}} \ \vee_{n \leq c}[c \doteq \nu(n)]$$
$$= [c \doteq \nu(0) \ \vee \ c \doteq \nu(1) \ \vee \ \ldots \ \vee \ c \doteq \nu(c)] : \mathbb{N} \to \mathbb{2}.$$

$\nu : \mathbb{N} \to \mathbb{N}^{+} \subset \mathbb{N}$ has codomain restriction

$$\nu : \mathbb{N} \to \dot{\mathbb{N}} \ =_{\mathrm{def}} \ \nu\mathbb{N} \ =_{\mathrm{def}} \ \{\mathbb{N} | \chi\nu\}$$

and is then an iso with PR inverse

$$\nu^{-1} = \nu^{-1}(c) \ =_{\mathrm{def}} \ \min_{n \leq c}[\nu(n) \doteq c] : \nu\mathbb{N} \xrightarrow{\cong} \mathbb{N}.$$

For a **PR**-map $f : \mathbb{N} \to \mathbb{N}$ **define** its *numeral twin*

$$\dot{f} \ =_{\mathrm{def}} \ \nu \circ f \circ \nu^{-1} : \dot{\mathbb{N}} = \nu\mathbb{N} \xrightarrow{\nu^{-1}} \mathbb{N} \xrightarrow{f} \mathbb{N} \xrightarrow{\nu} \dot{\mathbb{N}},$$

giving trivially (local) *naturality*

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "f"] \arrow[d, "\nu", "\cong"'] \arrow[dr, phantom, "="] & \mathbb{N} \arrow[d, "\nu", "\cong"'] \\
\dot{\mathbb{N}} \arrow[r, "\dot{f}"] & \dot{\mathbb{N}}
\end{tikzcd}$$

Extension of numeral sets and numeralisation to all Objects of **PR** (and of **PRa** :)

- $\dot{\mathbb{1}} = \{\dot{0}\} = \{\nu(0)\} = \{\ulcorner 0 \urcorner\} \subset \dot{\mathbb{N}} \subset \mathbb{N},$
  $\nu_{\mathbb{1}}(0) = \nu(0) : \mathbb{1} \xrightarrow{\cong} \dot{\mathbb{1}} \ =_{\mathrm{def}} \ \{\dot{0}\} \xrightarrow{\subseteq} \dot{\mathbb{N}}.$

- Recursive extension to products:

$$\frac{A, B \text{ in } \mathbf{PR}}{\begin{aligned}&\mathrm{dot}(A \times B) = \nu(A \times B) = \langle \dot{A} \dot{\times} \dot{B} \rangle \\ &=_{\mathrm{def}} \ \{\langle \nu_A(a), \nu_B(b) \rangle \,|\, a \in A, b \in B\} \\ &=_{\text{by def}} \ \{\mathbb{N} \times \mathbb{N} \,|\, (\nu_A \wedge \nu_B).\}\end{aligned}}$$

- Extension to subsets:

$$\frac{\chi = \chi(a) : A \to \mathbb{N} \text{ predicate}}{\begin{aligned}&\mathrm{dot}(\{A|\chi\}) = \{\dot{A}|\dot{\chi}\}\\ &= \{\nu(a) \,|\, a \in \{A|\chi\}\} \subseteq \nu A\end{aligned}}$$

- ν isomorphy and naturality extended to A, B in **PR** (**PRa**):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\cong"', "\nu_A"] \arrow[dr, phantom, "="] & B \arrow[d, "\cong"', "\nu_B"] \\
\dot{A} \arrow[r, "\dot{f}"] & \dot{B},
\end{tikzcd}$$

$$\dot{f} \ =_{\text{by def}} \ \nu_B \circ f \circ \nu_A^{-1} : \dot{A} \to A \to B \to \dot{B}.$$

Universal Objects $\mathbb{X}$, $\mathbb{X}_{\perp}$ of numerals and (nested) pairs of numerals:

Next **tool** needed is *coding* of all *(nested) pairs* of natural numbers, into a predicative, hence PR decidable *"Universal" Object* $\mathbb{X}$, enriched by a (second level) *undefined value* element, called $\perp : \mathbb{1} \to \mathbb{N}$, into set $\mathbb{X}_{\perp} = \mathbb{X} \,\dot{\cup}\, \{\perp\} \subset \mathbb{N}^{+}$, of binary bracketed NNO strings.

As code for *waste symbol* we take $\perp \ =_{\text{def}} \ \ulcorner\perp\urcorner \equiv \ulcorner\backslash\text{bot}\urcorner$.

[We will take $\perp$ as a *waste* instance for "values" of *defined* undefined arguments of *partial maps* with *predicatively defined* partiality.]

With this understanding of coded special symbols, we **define** PR the *Universal Objects*

$$\mathbb{X}, \mathbb{X}_{\perp} = \{\mathbb{N} \,|\, \mathbb{X}, \mathbb{X}_{\perp} : \mathbb{N} \to 2\} \subset \mathbb{N}$$

of all (codes of) *undefined value* $\perp$, natural number *numerals* $\dot{n} =_{\text{def}} \nu(n) \in \dot{\mathbb{N}}$, and (possibly nested) *pairs*

$$\langle x; y \rangle =_{\text{by def}} \ulcorner (\urcorner \, x \ulcorner , \urcorner \, y \ulcorner) \urcorner$$

of numerals as follows:

- the rôle of NNO in Universe Theories $\mathbf{PR}\mathbb{X}, \mathbf{PRa}\mathbb{X}$ below will be played by above (predicative) set $\dot{\mathbb{N}} = \nu\mathbb{N} = \{\mathbb{N}|\chi\nu\}$ of all internal *numerals,* **anchor** set for recursive **definition** of Universal Object $\mathbb{X} \subset \mathbb{N}$ PR **defined** by

  $\dot{\mathbb{N}} \subset \mathbb{X} \subset \mathbb{N}^+$, *numerals proper;* further recursively by

- $\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle =_{\text{def}} \{\langle x; y \rangle \,|\, x, y \in \mathbb{X}\} \subset \mathbb{X}$, (predicative) set of *(nested) pairs of numerals, general numerals,* in particular

$$\langle \mathbb{X} \dot{\times} \dot{\mathbb{N}} \rangle = \{\langle x; \dot{n} \rangle \,|\, x \in \mathbb{X}, n \in \mathbb{N}\} \subset \mathbb{X};$$

- $\mathbb{X}_{\perp} =_{\text{def}} \mathbb{X} \cup \{\perp\} \subset \mathbb{N}^+$.

This terminates recursive **definition** of ("minimal") predicative *Universal Objects* $\mathbb{X}$ and $\mathbb{X}_{\perp}$, of *nested pairs of numerals,* both

$$\mathbb{X},\ \mathbb{X}_{\perp} \subset \mathbb{N}^+ \equiv \mathbb{N}_{>0} \subset \mathbb{N}^* \equiv \mathbb{N}.$$

Remark: A *superUniversal Object* $\mathbb{U} \supset \mathbb{X}$, $\mathbb{U} \subset \mathbb{N}$ of *lists* (bracketed strings) of numerals can be **defined** PR by

- $\dot{\mathbb{N}} = \nu\mathbb{N} \subset \mathbb{U}$,
- $x \in \mathbb{U}, y \in \mathbb{U} \implies x; y \in \mathbb{U}$,
- $x \in \mathbb{U} \implies \langle x \rangle \in \mathbb{U}$.

Set $U \subset \mathbb{N}$ can be interpreted as set of (numeralised) coefficient lists $\mathbb{N}[x_1, x_2, \ldots, x_m, \ldots]$ of polynomials in *several indeterminates* $X_1, X_2, \ldots$ with (numeralised) coefficients out of $\mathbb{N}$, written in form $\mathbb{N}[X_1][X_2] \ldots$.

3.3 Universe Monoid $\mathbf{PR}\mathbb{X}$

Endomorphism <u>set</u> $\mathbf{PR}(\mathbb{N}, \mathbb{N}) \subset \mathbf{PR}$ is itself a **Monoid,** a categorical Theory with just one Object. The **PR** predicates/predicative subsets $\mathbb{X} \subset \mathbb{X}_{\perp} \subset \mathbb{N}$ give rise to the following

Embedded "Cartesian PR Monoid" $\mathbf{PR}\mathbb{X}$:

- the basic, only, "super" Object of $\mathbf{PR}\mathbb{X}$ is $\mathbb{X} \subset \mathbb{N}$, $\mathbb{X} : \mathbb{N} \to \mathbb{N}$ in $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ predicate/set of (internal) numerals and nested pairs of numerals.

- the rôle of NNO will be taken by its predicative subset—**PR** predicate
$$\dot{\mathbb{N}} = \nu\mathbb{N} = \{c \in \mathbb{N} \,|\, \chi\nu(c)\}$$
of internal *numerals* above. Here $\nu : \mathbb{N} \xrightarrow{\cong} \dot{\mathbb{N}} \subset \mathbb{N}$ is *numeralisation.*
$$\nu\mathbb{N} \subset \lceil \mathbb{1}, \mathbb{N} \rceil = \lceil \mathbb{1}, \mathbb{N} \rceil_{\mathbf{PR}} \subset \mathrm{PR} \subset \mathbb{N}$$
is part of set $\lceil \mathbb{1}, \mathbb{N} \rceil$ of internal map codes of **PR** "from" $\mathbb{1}$ "to" $\mathbb{N}$. That set internalises map <u>set</u> $\mathbf{PR}(\mathbb{1}, \mathbb{N})$.

- Further predicates are given PR by ν-product formation

$$\langle\dot{\times}\rangle \quad \frac{A, B\ \mathbf{PR}\mathbb{X}\ \text{Objects}}{\langle A \dot{\times} B\rangle\ \text{Object},}$$

"but" formally we consider explicitely in <u>Monoid</u> $\mathbf{PR}\mathbb{X}$ only predicates

$$\mathbb{X} \supset \dot{\mathbb{N}} \supset \dot{\mathbb{1}} \text{ as well as } \langle \mathbb{X} \dot{\times} \dot{\mathbb{N}} \rangle \subset \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \subset \mathbb{X}$$

as (proto) Objects.

- basic "universe" map constants, bå ∈ bås, are
  - "zero" $\dot{0} : \mathbb{N} \supset \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \supset \dot{\mathbb{1}} \ =_{\text{def}}\ \{\dot{0}\} \ =_{\text{def}}\ \{\nu(0)\} \ =_{\text{def}}$ $\{\ulcorner 0 \urcorner\} \xrightarrow{\subseteq} \dot{\mathbb{N}} \subset \mathbb{X}_{\underline{\perp}}$:
    $\mathbb{X} \ni \dot{0} \mapsto \dot{0} \in \dot{\mathbb{N}} \subset \mathbb{X}$,
    $\mathbb{X} \smallsetminus \dot{\mathbb{1}} \ni q \mapsto \underline{\perp} \in \mathbb{X}_{\underline{\perp}}$
  - "successor":
    $\dot{s} : \mathbb{N} \supset \mathbb{X} \supset \dot{\mathbb{N}} \ =_{\text{by def}}\ \{\mathbb{N} | \chi\nu\} = \{\dot{n} \in \mathbb{X} \,|\, n \in \mathbb{N}\} \to \dot{\mathbb{N}}$,
    $\dot{n} \mapsto \dot{s}\,\dot{n} \ =_{\text{def}}\ \nu(s\,n) \ =_{\text{by def}}\ \langle \ulcorner s \urcorner \odot \nu(n) \rangle$,
    $\mathbb{N} \smallsetminus \dot{\mathbb{N}} \ni q \mapsto \underline{\perp} \in \mathbb{X}_{\underline{\perp}}$.
    Symbolically: $\dot{s} = \langle \ulcorner s \urcorner \odot \_ \rangle : \dot{\mathbb{N}} \to \dot{\mathbb{N}} \subset \lceil \mathbb{1}, \mathbb{N} \rceil$.
    [From now on, we do not list any more the *trash* cases.]
  - "identity" $\mathrm{id} = \mathrm{id}_{\mathbb{X}} : \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \to \mathbb{X} \subset \mathbb{X}_{\underline{\perp}}$
    $\mathbb{X} \ni x \mapsto x \in \mathbb{X}$,

– "terminal map" $\mathring{\Pi} : \mathbb{X} \to \dot{\mathbb{1}} \subset \mathbb{X}$,
$\mathbb{X} \ni x \mapsto \dot{0} \in \{\dot{0}\} \subset \mathbb{X}$,

– "transposition" $\mathring{\Theta} : \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$,
$\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \ni \langle x; y \rangle \mapsto \langle y; x \rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$,
Here $\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \ =_{\text{def}} \ \{ \langle x; y \rangle \,|\, x \in \mathbb{X}, y \in \mathbb{X} \} : \mathbb{N} \supset \mathbb{X} \to \mathbb{N}$.

– "diagonal" $\mathring{\Delta} : \mathbb{X} \to \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$,
$\mathbb{X} \ni x \mapsto \langle x; x \rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$,

– "left projection" $\mathring{\ell} : \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \mathbb{X}$,
$\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \ni \langle x; y \rangle \mapsto x \in \mathbb{X}$,

– "right projection" $\mathring{r} \ =_{\text{def}} \ \mathring{\ell} \circ \Theta : \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \mathbb{X}, \langle x; y \rangle \mapsto y$.

- we close <u>Monoid</u> **PR**$\mathbb{X}$ under composition of Theorie **PR** :

$$(\circ) \quad \frac{f, g : \mathbb{N} \supset \mathbb{X} \to \mathbb{X} \text{ in } \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})}{g \circ f \text{ in } \mathbf{PR}\mathbb{X}.}$$

- closure under "cartesian" product of maps:

$$(\times) \quad \frac{f, g : \mathbb{X} \to \mathbb{X} \text{ in } \mathbf{PR}\mathbb{X}}{\begin{array}{l} \langle f \dot{\times} g \rangle : \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \to \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \text{ in } \mathbf{PR}\mathbb{X}, \\ \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \ni \langle x; y \rangle \mapsto \langle f\,x; g\,y \rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \subset \mathbb{X} \\ \text{recursively on the cartesian structure of } \mathbb{X}. \end{array}}$$

- "induced map":

$$(\mathrm{ind}) \quad \frac{f, g : \mathbb{X} \to \mathbb{X} \text{ in } \mathbf{PR}\mathbb{X}}{\begin{array}{l} \langle f \,.\, g \rangle : \mathbb{X} \to \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \text{ in } \mathbf{PR}\mathbb{X}, \\ \mathbb{X} \ni x \mapsto \langle f\,x; g\,x \rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \subset \mathbb{X}, \end{array}}$$

- "iterated" (formally interesting, see last line):

$$(\mathrm{it}) \quad \frac{f : \mathbb{X} \to \mathbb{X} \ \mathbf{PR}\mathbb{X} \text{ map}}{\begin{array}{l} f^{\dot{\S}} : \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \dot{\mathbb{N}} \rangle \to \mathbb{X} \text{ in } \mathbf{PR}\mathbb{X}, \\ \langle \mathbb{X} \dot{\times} \dot{\mathbb{N}} \rangle \ni \langle x; \dot{n} \rangle \mapsto f^n(x) \in \mathbb{X}, \\ n = \nu^{-1}(\dot{n}),\ \dot{n} \in \dot{\mathbb{N}} = \nu\mathbb{N} \ =_{\text{by def}} \ \{\mathbb{N} | \chi\nu\} \text{ free.} \end{array}}$$

- Notion of **map equality** for Theory $\mathbf{PR}\mathbb{X}$ is inherited from $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ i. e. from Theory $\mathbf{PR}$.

$\mathbf{PR}\mathbb{X}$ contains as maps all inclusions of $\mathbf{PR}$ Objects into $\mathbb{X}$, and their retractions:

$$\frac{A \text{ a } \mathbf{PR} \text{ Object (a finite power of } \mathbb{N})}{\begin{array}{l} \dot{A} \overset{\subset}{\to} \mathbb{X} \ \mathbf{PR}\mathbb{X} \text{ map,} \\ \mathrm{re}_A = \mathrm{re}_A(x) : \mathbb{X} \to \dot{A} \text{ in } \mathbf{PR}\mathbb{X}, \text{ where} \\ \mathbb{X} \ni \nu A(a) \mapsto \nu A(a) \in \dot{A}, \\ \mathbb{X} \setminus \dot{A} \ni x \mapsto \underline{\perp} \in \mathbb{X}_{\underline{\perp}}\,. \end{array}}$$

PR$\mathbb{X}$ contains all ν-twins $\mathring{f} : \mathbb{X} \to \mathbb{X}$ of **PR**-maps, **PR** diagram

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\mathring{f}"] \arrow[d, "\mathrm{re}_A"] \arrow[dr, phantom, "=_{\mathrm{def}}"] & \mathbb{X} \arrow[r, "\mathring{g}"] \arrow[d, bend left, "\mathrm{re}_B"] \arrow[dr, phantom, "="] & \mathbb{X} \\
\dot{A} \arrow[r, "\dot{f}"] & \dot{B} \arrow[r, "\mathring{g}"] \arrow[u, bend left, "\subset"] & \dot{C} \arrow[u, "\subset"']
\end{tikzcd}$$

PR$\mathbb{X}$ **Structure Theorem:** With the proto Objects, constants, maps, composition above,

- $\dot{\mathbb{1}}$ taken as "terminal Object",

  further proto Objects $\mathbb{X}, \mathbb{N}, \langle \mathbb{X} \dot{\times} \mathbb{N} \rangle, \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$,

- $\mathring{\Pi} : \mathbb{X} \to \dot{\mathbb{1}}$ taken as "terminal map,"

- "product" taken

  $$\begin{array}{l} \big\langle \ell : \langle A \dot{\times} B \rangle \to A : \langle x; y \rangle \to x, \\ r : \langle A \dot{\times} B \rangle \to B, \langle x; y \rangle \to y \big\rangle, \end{array}$$

- $\langle f\,.\,g\rangle : C \to \langle A \dot{\times} B\rangle$ taken as "induced map,"

  $x \mapsto \langle f\,x; g\,x\rangle$,

- $\langle \dot{\mathbb{1}} \xrightarrow{\dot{0}} \dot{\mathbb{N}} \xrightarrow{\dot{s}} \dot{\mathbb{N}}\rangle$ taken as NNO,

- and $f^{\dot{\S}} : \langle \mathbb{X} \dot{\times} \dot{\mathbb{N}}\rangle \to \mathbb{X}$ as iterated of

  PR$\mathbb{X}$ endomap $f : \mathbb{X} \to \mathbb{X}$, $\langle x; \nu n\rangle \mapsto f^n(x)$,

PR$\mathbb{X}$ becomes a proto Cartesian PR category (just lack of Objects).

• Fundamental Theory **PR** is naturally embedded into Theory **PR**$\mathbb{X}$, by faithfull Functor **I** say.

Proof as (formal) **Exercise:** Use in particular **Equality Definability** within **PR** for establishing the various universal properties within Theory **PR**$\mathbb{X}$, e.g. of terminal Object $\dot{\mathbb{1}} = \{\dot{0}\}$ (Question of Joseph).

3.4 Typed Universe Theory PRa$\mathbb{X}$

We now have at our disposal all formal means—in particular Universal predicative (sub)Object $\mathbb{X} \subset \mathbb{N}$ of singletons and nested $\mathbb{N}$-pairs as well as "type-free" Universe Monoid **PR**$\mathbb{X}$—for construction of a (Universe) Theory

$$\mathbf{PRa}\mathbb{X} \subseteq \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N}) \quad (\underline{\text{inclusions}} \text{ of map } \underline{\text{sets}}),$$

PRa $\sqsubset$ **PRa**$\mathbb{X}$ Cartesian PR embedding, of partial PR maps with *defined partiality,* this partiality now being given by *case distinction* on *predicate* $\chi(a) : A \to \mathbb{2}$ *selecting* those arguments $a \in A \cong \mathbb{N}^{\underline{m}}$ on which (all) formally partial **PRa** maps

$$f : A \supseteq \{A \,|\, \chi\} \to \{B \,|\, \psi\} \subseteq B$$

are *defined.*

We <u>embed</u> the maps of Theory **PRa** into Universe Theory **PR**$\mathbb{X} \subset$ **PR**$(\mathbb{N}, \mathbb{N})$. Theory **PR**$\mathbb{X}$ is to contain Theory **PRa**$\mathbb{X}$ as map-"sub" Category, using the outside (formal) *trash* Object $\{\underline{\perp}\} \subset \mathbb{X}_{\underline{\perp}} \cong \mathbb{X} \,\dot{\cup}\, \{\underline{\perp}\}$ for introduction of *defined partiality:*

$\underline{\perp}$ is interpreted as *defined undefined value* for *defined partial* PR-maps, these defined partial PR maps introduced originally in terms of **PR** maps $f : A \to B$: Theory **PRa** = **PR** + (abstr).

Following the general lines of Dana Scott for definition of partial maps, this *special* case of—*defined,* PR *decided*—partiality of maps is within this alternative frame **PRa**$\mathbb{X}$ defined by Case Distinction on (arguments out of) PR decided *Domain of definition* $\{A \,|\, \chi\} \subseteq A \equiv \{A \,|\, \mathrm{true}_A\}$ of *defined*-partial map

$$f : A \supseteq \{A \,|\, \chi\} \to \{B \,|\, \psi\} \subseteq B.$$

This alternative <u>presentation</u>

$$\mathbf{PRa} \cong \mathbf{PRa}^{\mathbb{X}} =_{\mathrm{def}} \mathbf{I}[\mathbf{PRa}] \sqsubset \mathbf{PRa}\mathbb{X} \subset \mathbf{PR}(\mathbb{X}, \mathbb{X})$$

of **PRa** is—in many aspects—technically easier to handle than original **PRa** with its extra—formally coarser—notion $f =^{\mathbf{PRa}} g : \{A \,|\, \chi\} \to \{B \,|\, \psi\}$ of equality.

There is an analogy with elegant definition of integers Domain $\mathbb{Z}$ as Quotient Object $(\mathbb{N} \times \mathbb{N})/\dot{=}^{\mathbb{Z}}$ versus direct definition $\mathbb{Z} =_{\mathrm{def}}$

$\mathbb{N}_{>0} \dot{\cup} \{0\} \dot{\cup} \mathbb{N}_{>0}$. This latter $\mathbb{Z}$ inherits $\mathbb{N}$'s notion of equality on all of its components, and in this approach $\mathbb{Z}$'s algebraic and order structure are all defined "directly" via case distinction, cf. Sarorius 1981.

For construction of Theory $\mathbf{PRa}\mathbb{X}$ we rely on

Equality Enumeration: As any theory here, *fundamental* Theory **PR** of Primitive Recursion as well as *basic* Theory $\mathbf{PRa} = \mathbf{PR} +$ (abstr)—definitional enrichment of **PR** by the schema of *predicate abstraction* $\langle \chi \rangle \mapsto \{A \,|\, \chi\}$, a "virtual", *abstracted* Object in **PRa**—admit an (external) primitive recursive enumeration of their respective **theorems,** ordered by length (more precisely: by lexicographical order) of the first **proofs** of these (equational) Theorems, here:

$$=^{\mathbf{PR}} (\underline{k}) : \underline{\mathbb{N}} \to \mathbf{PR} \times \mathbf{PR} \subset \underline{\mathbb{N}} \times \underline{\mathbb{N}} \text{ and}$$
$$=^{\mathbf{PRa}} (\underline{k}) : \underline{\mathbb{N}} \to \mathbf{PRa} \times \mathbf{PRa} \subset \underline{\mathbb{N}} \times \underline{\mathbb{N}}$$

respectively.

By the PR Representation Theorem 5.3 of Romàn 1989, these enumerations give rise to their internal versions—*formalisations*—

$$\check{=}^{\mathbf{PR}}_{k} : \mathbb{N} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N}^2 \text{ and}$$
$$\check{=}^{\mathbf{PRa}}_{k} : \mathbb{N} \to \mathrm{PRa} \times \mathrm{PRa} \subset \mathbb{N}^2,$$

with internalisation (*representation*) property

$$\mathbf{PR} \vdash \ \check{=}_{\mathrm{num}(\underline{k})} \ = \ \mathrm{num}(=^{\mathbf{PR}}_{\underline{k}}) : \mathbb{1} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N}^2 \text{ and}$$
$$\mathbf{PRa} \vdash \ \check{=}^{\mathbf{PRa}}_{\mathrm{num}(\underline{k})} \ = \ \mathrm{num}(=^{\mathbf{PRa}}_{\underline{k}}) : \mathbb{1} \to \mathrm{PRa} \times \mathrm{PRa} \subset \mathbb{N}^2.$$

Here (external) <u>numeralisation</u> is given externally <u>PR</u> as

$$\mathrm{num}(\underline{n}) = s^{\underline{n}} \circ 0 : \mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \ldots \xrightarrow{s} \mathbb{N},$$
$$\mathrm{num}(\underline{m}, \underline{n}) = (\mathrm{num}(\underline{m}), \mathrm{num}(\underline{n})) : \mathbb{1} \to \mathbb{N} \times \mathbb{N},$$
$$\underline{m}, \underline{n} \text{ (“meta”) } \underline{\text{free}} \text{ in } \underline{\mathbb{N}}.$$

$\mathrm{PR} = \{\mathbb{N} \,|\, \mathrm{PR}\}$ is the predicative, PR decidable subset of $\mathbb{N}$ “of all **PR** codes”, *internalisation* of $\mathbf{PR} \subset \underline{\mathbb{N}}$ of all **PR**-<u>terms</u> on Object Language level. Analogous meaning for *internalisation* $\mathrm{PRa} \subset \mathbb{N}$ of $\mathbf{PRa} \subset \underline{\mathbb{N}}$, and for **PR**$\mathbb{X}$'s internal map (code) set $\mathrm{PR}\mathbb{X} \subset \mathbb{N}$.

In particular for later discussion of constructive (!) evaluation, we will need representation of all **PR** and **PRa** maps within Universe <u>Monoid</u> $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$.

We **define,** within endo map <u>set</u> $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ subTheories $\mathbf{PR}^{\mathbb{X}}$ as well as **PRa**$\mathbb{X}$—to become **isomorphic** to **PR** and to **extend PRa** respectively—externally PR as follows, by mimikry of schema (abstr), but *without* introduction of a coarser notion of equality as in case of schema of abstraction constituting Theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$.

So Theory $\mathbf{PRa}\mathbb{X} \sqsupset \mathbf{PR}\mathbb{X}$, map code set $\mathrm{PRa}\mathbb{X} = \mathrm{PR}\mathbb{X} \subset \mathbb{N}$ comes in, by external PR <u>enumeration</u> of its Object and map <u>terms</u> as follows:

Objects of **PR**$\mathbb{X}$ are—recall—the **PR**-predicates

$\mathbb{X} : \mathbb{N} \to \mathbb{N}$ *nested pairs of numerals,*

$$\mathbf{I}\,\mathbb{N} = \dot{\mathbb{N}} \;=_{\text{by def}}\; \{\dot{n} \,|\, n \in \mathbb{N}\} \;=_{\text{by def}}\; \{\mathbb{N} \,|\, \chi\nu\} : \mathbb{N} \to \mathbb{N},$$
$$\mathbf{I}\,\mathbb{1} = \dot{\mathbb{1}} \;=_{\text{by def}}\; \{\dot{0}\} : \mathbb{N} \to \mathbb{N},$$

and further recursively

$$\langle A \dot{\times} B \rangle \;=_{\text{by def}}\; \{\langle a; b\rangle \,|\, a \in A,\ b \in B\} \subset \mathbb{X}.$$

PR$\mathbb{X}$-maps in **PR**$\mathbb{X}(A, B)$ are **PR**$\mathbb{X}$-maps $f : \mathbb{X} \to \mathbb{X}$ such that

$$\mathbf{PR} \vdash\ A(x) \implies B(f(a)) \text{ and}$$
$$\mathbf{PR} \vdash\ \neg A(x) \implies f(x) \doteq \underline{\perp}\,.$$

Objects of **PRa**$\mathbb{X}$ are "predicates" $\{\mathbb{X} \,|\, \chi : \mathbb{X} \to \dot{\mathbb{N}}\}$, in particular those of form $\{\dot{A}|\dot{\chi}\}$, $\{A|\chi\}$ **PRa** Object, $\dot{\chi}\ =_{\text{def}}\ \mathbf{I}\,\chi : \mathbb{N} \to \mathbb{N}$ (in **PR**.)

PRa$\mathbb{X}$ maps from Object $\{A|\chi\}$ to $\{B|\psi\}$ are **PR**$\mathbb{X}$-maps f : $\mathbb{X} \to \mathbb{X}$ such that

$$\mathbf{PR} \vdash\ \chi(a) \implies \psi\, f(a) \text{ and}$$
$$\mathbf{PR} \vdash\ \neg\,\chi(a) \implies f(a) \doteq \underline{\perp}\,.$$

Observe the **truncated** parallelism to **definition** of **PRa**-maps

$$f : \{A \,|\, \chi\} \to \{B \,|\psi\}.$$

This given, Embeddding Functor

$$\mathbf{I} : \mathbf{PR} \sqsubset \mathbf{PR}\mathbb{X}$$

is **defined** <u>PR</u> as follows:

$$\mathbf{I}\,\mathbb{1} = \dot{\mathbb{1}}\ =_{\text{by def}}\ \{\dot{0}\} : \mathbb{X} \to \mathbb{N}$$
$$\mathbf{I}\,\mathbb{N} = \dot{\mathbb{N}} = \nu\mathbb{N}\ =_{\text{def}}\ \{\dot{n} \,|\, n \in \mathbb{N}\}\ =_{\text{by def}}\ \{\mathbb{N} \,|\, \chi\nu\} : \mathbb{X} \to \mathbb{N},$$

and further <u>recursively</u> :

$$\mathbf{I}\,(A \times B)\ =_{\text{def}}\ \langle \mathbf{I}\,A \,\dot{\times}\, \mathbf{I}\,B \rangle$$
$$=_{\text{by def}}\ \{\langle a; b\rangle \,|\, a \in \dot{A}, b \in \dot{B}\} : \mathbb{X} \to \mathbb{N}.$$

Functorial **definition** of **I** on **PR** maps:

$$\mathbf{PR}(A,B) \ni f \overset{\mathbf{I}}{\mapsto} \mathbf{I}\,f = \dot{f} \in \mathbf{PR}\mathbb{X}$$

then is "canonical", by external PR on the structure of **PR**-map $f : A \to B$, in particular by mapping all "arguments" in $\mathbb{N} \smallsetminus \dot{A} = \mathbb{N} \smallsetminus \mathbf{I}\,A$ into $\underline{\perp} \in \mathbb{X}_{\underline{\perp}}$: one *waste basket* outside all Objects of **PR**$\mathbb{X}$.

$\dot{f}$ is the restriction of $\ulcorner f \urcorner \odot \_ = \lceil \mathbb{1}, \ulcorner f \urcorner \rceil : \lceil \mathbb{1}, A \rceil \to \lceil \mathbb{1}, B \rceil$ to νA and νB.

Interesting now is that we can extend embedding **I** above into an <u>embedding</u> $\mathbf{I} : \mathbf{PRa} \longrightarrow \mathbf{PRa}\mathbb{X} \sqsupset \mathbf{PR}\mathbb{X}$, by the following

Definition: For a (general) **PRa** Object, of form $\{A \,|\, \chi\} : A \to \mathbb{2}$, define

$$\mathbf{I}\,\{A \,|\, \chi\}(x) \ =_{\mathrm{def}} \ \{x \in \dot{A} \,|\, \dot{\chi}\,(x) \doteq \dot{1}\} = \{x \in \mathbf{I}\,A \,|\, \mathbf{I}\,\chi\,(x) \doteq \mathbf{I}\,\mathrm{true}\}$$

Special case $\chi = \mathbb{X}$:

$$\dot{\mathbb{X}} = \mathbf{I}\,\mathbb{X} = \{n \in \dot{\mathbb{N}} \,|\, \dot{\mathbb{X}}(n) \doteq \dot{1}\} = \{\langle x \rangle \in \dot{\mathbb{N}} \,|\, x \in \mathbb{X}\} :$$

Members of $\dot{\mathbb{X}} \subset \mathbb{N}$ get double outmost angle brackets.

Define now embedding **I** on **PRa** *maps* $f : \{A \,|\, \chi\} \to \{B \,|\, \psi\}$ by

$$\mathbf{I}\,f\,(x) \ =_{\mathrm{def}} \ \underline{\perp} \text{ if } x \not\in \mathbf{I}\,A,$$

$$\mathbf{I}\,f\,(\dot{a}) \ =_{\mathrm{def}} \ \begin{cases} \mathrm{dot}(f(a)) \text{ if } \chi(a) \doteq \mathrm{true} \\ \underline{\perp} \ \text{ if } \chi(a) \doteq \mathrm{false} \end{cases}$$

$$: \mathbb{X}_{\underline{\perp}} \supset \mathbf{I}\,\{A \,|\, \chi\} \to \mathbf{I}\,\{B \,|\, \psi\} \,\dot{\cup}\, \{\underline{\perp}\} \subset \mathbb{X}_{\underline{\perp}}$$

Here $\dot{a} \in \mathbb{X}$ is **defined** recursively by

$$\dot{n} = \nu(n) \text{ for } n \in \mathbb{N} \text{ (see above)},$$

$$\mathrm{dot}((a,b)) = \langle \dot{a}; \dot{b} \rangle : \ \text{replace brackets ( ) by } \langle\ \rangle.$$

Comment: We replace "don't-worry arguments" in the complement $\neg\,\chi$ of **PRa** Object $\{A\,|\,\chi\}$ by *cutting* them *out* in the definition of *replacing* $\mathbf{PR}\mathbb{X}$ Object $\mathbf{I}\,\{A\,|\,\chi\}$. "Coarser" notion $=^{\mathbf{PRa}}$ (coarser then $=^{\mathbf{PR}}$) is then replaced by original notion of equality, $=^{\mathbf{PR}}$ itself, notion of map-equality of *roof* $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$:

This formal "sameness" with PR equality was the goal of the considerations above: The new theory version $\mathbf{PRa}\mathbb{X}$ is a **sub**Theory of **PR** with *notion of equality*—objectively as well as (then) *internally*—inherited from *fundamental* Theory **PR**, *and* it comes with a universal Object, $\mathbb{X}$: One Object for all in evaluation.

From the above we obtain **straightforward**

Universal Embedding Theorem:

(i) $\mathbf{I} : \mathbf{PR} \longrightarrow \mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$ above is a faithfull Functor.

(ii) Theory $\mathbf{PRa}\mathbb{X}$ "inherits" from category **PRa** all of its (categorically described) structure: Cartesian PR category structure, equality predicates on all Objects, schema of predicate abstraction, equalisers, and—trivially—the whole algebraic, logic and order structure on NNO $\dot{\mathbb{N}}$ and truth Object $\dot{2}$.

(iii) **PR** map embedding **I** "canonically" extends into a Cartesian PR functorial <u>embedding</u> (!)

$$\mathbf{I} : \mathbf{PRa} \longrightarrow \mathbf{PRa}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$$

<u>of</u> Theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ <u>to</u> *Universe Theory* $\mathbf{PRa}\mathbb{X}$ *with predicate abstraction.*

(iv) Embedding **I** above **defines** a PR isomorphism of categories

$$\mathbf{I} : \mathbf{PRa} \xrightarrow{\cong} \mathbf{PRa}^{\mathbb{X}} =_{\mathrm{def}} \mathbf{I}[\mathbf{PRa}] \sqsubset \mathbf{PRa}\mathbb{X}.$$

(v) isomorphism $\nu : \mathbb{N} \overset{\cong}{\to} \dot{\mathbb{N}} = \nu\mathbb{N}$ extends to natural isomorphism family $\nu_A : A \overset{\cong}{\to} \dot{A} = \nu A$, A in **PR**, **PRa**, transformation ν given recursively by

$$\begin{aligned}
&\nu_{\mathbb{N}} = \nu, \\
&\nu(A \times B) = \langle \nu A \times \nu B \rangle \\
&\ = \{\langle a; b\rangle \in \mathbb{X} \mid \chi\nu A(a) \wedge \chi\nu B(b)\}, \\
&\nu(A \times B)(a, b) = \langle \nu A(a); \nu B(b)\rangle \in \nu(A \times B) \subset \mathbb{X}.
\end{aligned}$$

(vi) we put things together into the following diagram:

$$\begin{tikzcd}
 & \{A \mid \chi\} \arrow[r, "f"] \arrow[d, "\nu\{A \mid \chi\}"', "\cong"] \arrow[dr, phantom, "="] & \{B \mid \psi\} \arrow[d, "\cong"', "\nu\{B \mid \psi\}"] & \\
\{\dot{A} \mid \dot{\chi}\} \arrow[r, equal] & \mathbf{I}\{A \mid \chi\} \arrow[r, "\mathbf{I} f"] \arrow[d, "\subset"] & \mathbf{I}\{B \mid \psi\} \arrow[r, "\subset"] & \mathbf{I}\{B \mid \psi\} \dot{\cup} \{\perp\} \arrow[d, "\subset"] \\
 & \mathbb{X}_{\perp} \arrow[rr, "\dot{f} =_{\mathrm{by\,def}} \mathbf{I}_{\mathbf{PR}} f"] \arrow[d, "\subset"] \arrow[drr, phantom, "="] & & \mathbb{X}_{\perp} \arrow[d, "\subset"] \\
 & \mathbb{N} \arrow[rr, "\dot{f}"] & & \mathbb{N}
\end{tikzcd}$$

PRa Embedding DIAGRAM for $\mathbf{I}\, f$ **q.e.d.**

Remark: As frame for evaluation we will take original PR Theory **PRa** with predicate abstraction, as well as its strengthening, PR

Descent Theory $\pi\mathbf{R}$. this will in particular avoid formal conflict—on frame niveau—between $(\ldots)$ and $\langle\ldots\rangle$, comma and semicolon etc.

Another possiblity would have been change of notation for Theory $\mathbf{PRa}\mathbb{X}$: That is logically more complex. It would result in conservatively adding a new (again countable) Universal Object, $\bar{\mathbb{X}}$ say, to Theory $\mathbf{PRa}$, formally (!) containing all Objects of $\mathbf{PR}$ (and then of $\mathbf{PRa}$), and generating the whole Cartesian PR theory structure with abstraction into an embedding theory, $\mathbf{PR\bar{a}}$ say.

Chapter 4

Evaluation

4.1 Descent Axiom Schema

Gödel's first Incompleteness Theorem for *Principia Mathematica und verwandte Systeme I,* on which in particular is based the second one, on non-prova*bility* of **PM**'s own *Consistency formula* $\mathrm{Con}_{\mathbf{PM}}$, exhibits a (closed) **PM** formula φ with property

$$\mathbf{PM} \vdash \ [\varphi \iff \neg\,(\exists\, k \in \mathbb{N})\,\mathrm{Prov}_{\mathbf{PM}}(k,\, \ulcorner \varphi \urcorner\,)\,], \text{ in words:}$$

Theory **PM** derives φ to be equivalent to its "own" *coded, arithmetised non-Provability.*

This equivalence needs—already for its *statement*—*full* formal, "not testable" quantification. So, **first** the *Consistency Provability* issue is not settled for—constructive—*Free-Variables Primitive Recursive Arithmetics* **PRa** and **PRa**$\mathbb{X}$ (the latter with inbuilt *Universal Object* $\mathbb{X}$ and defined partiality given by case distinction), as well as for their strengthenings, theories $\mathbf{S}$ say, which nevertheless express (formalised, internal) *Consistency* as Free-Variable—FV—formula

$$\mathrm{Con}_{\mathbf{S}} = \mathrm{Con}_{\mathbf{S}}(k) = \neg\,\mathrm{Prov}_{\mathbf{S}}(k,\ \ulcorner\mathrm{false}\urcorner\,) : \mathbb{N} \to \mathbb{2} :$$

"No $k \in \mathbb{N}$ is a *Proof* code *proving* $\ulcorner$false$\urcorner$."

This is the point of depart for investigation of a "suitable" strengthening $\pi\mathbf{R} = \mathbf{PRa} + (\pi)$ of categorical Universe PR Theory $\mathbf{PRa}$ of Primitive Recursion, with its *predicate abstraction Objects* $\{A \,|\, \chi\} = \{a \in A \,|\, \chi(a)\}$.

We rely on **axiom schema** of *non-infinite iterative Descent*

$$(\pi)\quad \frac{\begin{array}{l} c = c\,(a) : A \to \mathbb{N}[\omega] \text{ PR } (\textit{complexity}), \\ p = p(a) : A \to A \text{ PR } (\textit{predecessor endo}), \\ c\,(a) > 0 \implies c\,p(a) < c\,(a) \ (\textit{descent}), \\ c\,(a) \doteq 0 \implies p(a) \doteq a \ (\textit{stationarity}) \\ \psi = \psi(a) : A \to \mathbb{2} \ (\textit{"negative" test predicate}), \\ \psi(a) \implies c\,p^n(a) > 0 \ (\text{"all } n\text{"}, \textit{ to be excluded}) \end{array}}{\psi(a) = \mathrm{false}_A(a) : A \to \mathbb{2}.}$$

[The first four lines of the *antecedent* constitute (p, c) as (the data of) a CCI : of a *Complexity Controlled Iteration,* with (stepwise) descending order values in—polynomial—*Ordinal* $\mathbb{N}[\omega] \subset \mathbb{N}^*$ ordered lexicographically. Central **example:** *General Recursive,* ACKERMANN type *PR-code evaluation ev* will be *resolved* into such a CCI]

The point is is that (π) expresses an **axiom** which "we all" **believe** in (and which is a **Theorem** in **set** theory): Nobody has pointed to—will be able (?) to point to—any *infinitely descending chain* in $\mathbb{N}[\omega] \ =_{\text{by def}} \ \mathbb{N}^+ \subset \mathbb{N}^*$ (provided with its lexicographical order), a fortiori *not* to an *iterative* such: to an infinitely descending CCI.

Dangerous bound: is there a good reason that evaluation gives not a (code-) *self-evaluation* for Theory $\pi\mathbf{R}$?

Answer: *ev* is—by **definition**—*not* PR: If you take the *diagonal*

$$\text{diag}(n) \ =_{\text{def}} \ ev\,(\text{enum}_{\text{PR}}(n), \text{cantor}_{\mathbb{X}}(n)) : \mathbb{N} \to \mathbb{N},$$

enum_{PR} an internal PR *count* of all PR map codes, and $\text{cantor}_{\mathbb{X}} : \mathbb{N} \xrightarrow{\cong} \mathbb{X}$ "the" Cantor's *count* of $\mathbb{X} \subset \mathbb{N}$, then you get an Ackermann diagonal function[1] which grows faster than any PR function: but $\pi\mathbf{R}$ has only **PRa** maps as its *maps,* it is a (pure) *strengthening* of **PRa**.

On the other hand, *ev* is *intuitively* total, since—intuitively—complexity $c\,e^m(u,x) \in \mathbb{N}[\omega]$ "must" reach *minimum* $0 \in \mathbb{N}[\omega]$ in *finitely many* evaluation steps e (to come.) The latter intuition can be, in Free Variables, expressed *formally* by $\pi\mathbf{R}$'s schema (π).

Schema (π) says in fact that a condition which implies *infinite descent* of such a chain (on all x), must be *false* (on all x), since implied infinite descent is to be *excluded,* just by (plausible) **axiom.**

That a CCI does not descend infinitely—schema (π)—means categorically that (partially defined) while loop $\text{wh}[c > 0\,|\,p] : A \rightharpoonup A$

[1] for a two-parameter, simpler Ackermann function cf. Eilenberg & Elgot 1970

epi-terminates, i. e. has an epimorphic *defined-arguments enumeration*

$$d = d(a) = a : D_{\mathrm{wh}} = \{(a,n) \mid c\,p^n(a) \doteq 0 \in \mathbb{N}[\omega]\} \to A,$$
$$[\,\text{and rule } \widehat{\mathrm{wh}} = \widehat{\mathrm{wh}}(a,n) = p^n(a) : D_{\mathrm{wh}} \to A\,].$$

Within quantified Arithmetic ("**set** theory") **T** such epi is a retraction: the while loop terminates, on any argument $a \in A$. But it is not PR in general.

4.2 Iterative Code Evaluation

Building on forgoing chapter, we now **define** *coding* for *Universe PR Theory* **PR**$\mathbb{X}$ as well as (iterative) code *evaluations* $ev(u,x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$, all of this within (frame) Theory

$$\mathbf{PRa}\ [\,\cong \mathbf{PRa}^{\mathbb{X}} = \mathbf{I}[\mathbf{PRa}] \sqsubset \mathbf{PRa}\mathbb{X}\,],$$

see last forgoing chapter, or within **set** theory **T**.

[You may take frame **PRa** here as a—weak—**set** theory, and then check if **PRa** in fact is good for all of the constructions and properties needed—**set** theory is good for, as frame theory, as is easily seen in each particular case.]

Coding—*gödelisation*—is **defined** here just as *prime power product* coding of **ASCII** based LaTeX *source code items* into Object $\mathbb{N}^+ \equiv \mathbb{N}$ of non-empty strings of natural numbers—cf. coding above of nested pairs of numerals in(to) universal Object $\mathbb{X} \subset \mathbb{N}$.

For a *basic map constant*

$$\mathrm{b\mathring{a}} \in \mathrm{b\mathring{a}s} \ =_{\mathrm{def}}\ \{\mathrm{id} = \mathrm{id}_{\mathbb{X}}, \dot{0}, \dot{s}, \mathring{\Pi}, \mathring{\Theta}, \mathring{\Delta}, \mathring{\ell}, \mathring{r}\}$$

of **PR**$\mathbb{X}$ **define** *code* $\ulcorner$bå$\urcorner$: $\mathbb{1} \to \mathbb{N}$ to be the LaTeX-source-code of bå, in turn coded into $\mathbb{N}^+ \equiv \mathbb{N}_{>0}$ as a product of prime powers: quasi original *gödelisation,* into $\mathbb{N}^+ \subset \mathbb{N}$, the NNO of frame Theory **PRa**.

Exception: codes of $\dot{0}, \dot{s}$ are taken $\ulcorner 0 \urcorner$, $\ulcorner s \urcorner$, not $\ulcorner\ulcorner 0 \urcorner\urcorner$, $\ulcorner\ulcorner s \urcorner\urcorner$.

[Explicit coding of **PRa**$\mathbb{X}$ will not be needed for evaluation here]

Map composition $(g \circ f)$ then is (LaTeX) coded

$$\ulcorner(g \circ f)\urcorner \ =_{\mathrm{def}}\ \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \ =_{\mathrm{by\,def}}\ \langle \ulcorner g \urcorner \ \ulcorner \circ \urcorner \ \ulcorner f \urcorner \rangle,$$

Cartesian map product $\langle f \mathbin{\dot{\times}} g \rangle$ as

$$\ulcorner\langle f \mathbin{\dot{\times}} g \rangle\urcorner \ =_{\mathrm{def}}\ \langle \ulcorner f \urcorner \# \ulcorner g \urcorner \rangle \ =_{\mathrm{def}}\ \langle \ulcorner f \urcorner \ \ulcorner \dot{\times} \urcorner \ \ulcorner g \urcorner \rangle,$$

induced map $\langle f \,.\, g \rangle$ as

$$\ulcorner\langle f \,.\, g \rangle\urcorner \ =_{\mathrm{def}}\ \langle \ulcorner f \urcorner ; \ulcorner g \urcorner \rangle.$$

[Redundant, induced may be expressed via $\mathring{\Delta}$ and $\dot{\times}$.]

Iterated map $f^{\S} : \langle \mathbb{X} \times \nu\mathbb{N} \rangle \to \mathbb{X}$ is coded

$$\ulcorner f^{\S} \urcorner \ = \ \ulcorner f \urcorner^{\$} \ =_{\mathrm{by\,def}}\ \ulcorner f \urcorner^{\ulcorner \S \urcorner}.$$

Map code set PR$\mathbb{X} \subset \mathbb{N}^+ \equiv \mathbb{N}_{>0}$ of Theory **PR**$\mathbb{X}$ is PR **defined** by

- $\ulcorner$bå$\urcorner$ $\in$ PR$\mathbb{X}$ for bå $\in$ bås;
- $u, v \in \mathrm{PR}\mathbb{X} \implies \langle v \odot u \rangle \in \mathrm{PR}\mathbb{X}$,
- $u, v \in \mathrm{PR}\mathbb{X} \implies \langle u \# v \rangle \in \mathrm{PR}\mathbb{X}$,
- $u, v \in \mathrm{PR}\mathbb{X} \implies \langle u ; v \rangle \in \mathrm{PR}\mathbb{X}$,

- $u \in \mathrm{PR}\mathbb{X} \implies u^{\$} =_{\mathrm{def}} u^{\ulcorner\S\urcorner} \in \mathrm{PR}\mathbb{X}$.

- Formally, all of these (internal) codes have Dom and Codom $\mathbb{X}_{\underline{\perp}}$. The set $\mathrm{PR}\mathbb{X} \subset \mathbb{N}$ can be seen to contain the union of all the code sets $\lceil A, B \rceil$ "from A to B," A, B (proto) Objects of $\mathbf{PR}\mathbb{X}$.

For **Definition** of *evaluation ev* we first introduce *evaluation step* of form

$$e\,(u, x) = (e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x)) : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}},$$

by Primitive Recursion. This within "outer" Theory **PRa** which already has **PR** predicates $\mathbb{X}, \mathbb{X}_{\underline{\perp}}, \langle \mathbb{X} \times \nu\mathbb{N} \rangle : \mathbb{N} \to \mathbb{N}$ as Objects.

Comment: $e_{\mathrm{arg}}(u, x) \in \mathbb{X}_{\underline{\perp}}$ means here one-step u-evaluated *argument,* and $e_{\mathrm{map}}(u, x)$ designates the remaining part of *map code* u still to be evaluated after that evaluation step.

Notation: *Argument* $x \in \mathbb{X}$ in the below is to mean

- constant $\mathring{ba} \in \mathbb{X}$, short for $\mathbf{PR}\mathbb{X}$ map $\mathring{ba} \circ \mathring{\Pi} : \mathbb{X} \to \mathbb{1} \to \mathbb{X}$,

- free variable $x \in \mathbb{X}$, i. e. $x = \mathrm{id}_X$ or x a projection, in particular $x = \mathring{\ell} : \mathbb{X} \supset \langle \mathbb{X} \dot{\times} \nu\mathbb{N} \rangle \to \mathbb{X}$,

- x a *dependent variable,* i. e. a map <u>term</u> $h(y) : \mathbb{X} \to \mathbb{X}$, just a $\mathbf{PR}\mathbb{X}$ map. This case is the general one, it specialises to the other cases.

PR **Definition** of step e, PR on $\mathrm{depth}(u) \in \mathbb{N}$, now runs as follows:

- depth$(u) = 0$, i. e. u of form $\ulcorner$bå$\urcorner$, bå in bås one of the basic map constants of Theory $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}$:

$$e_{\mathrm{arg}}(\ulcorner \text{bå} \urcorner, x) =_{\mathrm{def}} \text{bå}(x),$$
$$e_{\mathrm{map}}(\ulcorner \text{bå} \urcorner, x) =_{\mathrm{def}} \ulcorner \mathrm{id} \urcorner,$$

the following are the

- cases of internal composition:

$$e\,(\langle v \odot \ulcorner \mathrm{id} \urcorner \rangle, x) =_{\mathrm{def}} (v, x),$$
and for $u \neq \ulcorner \mathrm{id} \urcorner$:
$$e\,(\langle v \odot u \rangle, x) =_{\mathrm{def}} (\langle v \odot e_{\mathrm{map}}(u, x) \rangle, e_{\mathrm{arg}}(u, x)) :$$
evaluate first map code u, on argument x.

- Cartesian cases:

$$e\,(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{id} \urcorner \rangle, \langle y; z \rangle) =_{\mathrm{def}} (\ulcorner \mathrm{id} \urcorner, \langle y; z \rangle),$$
a terminating case.
For $\langle u \# v \rangle \neq \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{id} \urcorner \rangle$:
$$e\,(\langle u \# v \rangle, \langle y; z \rangle)$$
$$=_{\mathrm{def}} (\langle e_{\mathrm{map}}(u, y) \# e_{\mathrm{map}}(v, z) \rangle, \langle e_{\mathrm{arg}}(u, y); e_{\mathrm{arg}}(v, z) \rangle),$$

evaluate u and v in parallel.

Here free variable x on $\mathbb{X}$ legitimately runs only on $\langle \mathbb{X} \dot{\times} \mathbb{X} \rangle \subset \mathbb{X}$, takes there the pair form $\langle y; z \rangle$. $x \in \mathbb{X} \smallsetminus \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$ results in present evaluation case into $\perp$.

- Cases of an induced (redundant via $\ulcorner\Delta\urcorner$ and $\odot$):

$$\begin{aligned}
&e\,(\langle\,\ulcorner\mathrm{id}\urcorner\,;\,\ulcorner\mathrm{id}\urcorner\,\rangle, z) \;=_{\mathrm{def}}\; (\,\ulcorner\mathrm{id}\urcorner\,, \langle z; z\rangle),\\
&\textit{a terminating}\ \text{case.}\\
&\text{For } \langle u; v\rangle \neq \langle\,\ulcorner\mathrm{id}\urcorner\,;\,\ulcorner\mathrm{id}\urcorner\,\rangle :\\
&e\,(\langle u; v\rangle, z)\\
&\quad =_{\mathrm{def}}\; (\langle e_{\mathrm{map}}(u,z); e_{\mathrm{map}}(v,z)\rangle, \langle e_{\mathrm{arg}}(u,z); e_{\mathrm{arg}}(v,z)\rangle),
\end{aligned}$$

  evaluate both components u and v.

- iteration case, with $\$:= \ulcorner\S\urcorner$ designating internal *iteration:*

$$\begin{aligned}
&e\,(u^{\$}, \langle y; \nu n\rangle) = (u^{[n]}, y) :\\
&\mathrm{PR}\mathbb{X} \times \mathbb{X} \supset \mathrm{PR}\mathbb{X} \times \langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \to \mathrm{PR}\mathbb{X} \times \mathbb{X}.
\end{aligned}$$

  Here $\nu n \in \nu\mathbb{N}$ free, $n := \nu^{-1}(\nu n) \in \mathbb{N}$, and $u^{[n]}$ is given by *code expansion* as

$$u^{[0]} \;=_{\mathrm{def}}\; \ulcorner\mathrm{id}\urcorner\,,\; u^{[n+1]} \;=_{\mathrm{def}}\; \langle u \odot u^{[n]}\rangle.$$

- trash case $e\,(u, x) = (\,\ulcorner\mathrm{id}\urcorner\,, \underline{\perp}\,) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}$ if (u, x) in none of the above—regular—cases.

For to convince ourselves on termination of iteration of step $e : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}$—on a pair of form $(\,\ulcorner\mathrm{id}\urcorner\,, x)$—we **introduce:**

(*Descending*) *complexity*

$$c_{ev}(u, x) = c\,(u) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \xrightarrow{\ell} \mathrm{PR}\mathbb{X} \to \mathbb{N}[\omega]$$

defined PR as

$$
\begin{aligned}
& c\,(\ulcorner\mathrm{id}\urcorner\,) \;=_{\mathrm{def}}\; 0 = 0 \cdot \omega \in \mathbb{N}[\omega], \\
& c\,(\ulcorner\mathrm{b\mathring{a}}'\urcorner\,) \;=_{\mathrm{def}}\; 1 \in \mathbb{N}[\omega] \\
& \quad \text{for } \mathrm{b\mathring{a}}' \text{ one of the other basic map constants in } \mathrm{b\mathring{a}s}, \\
& c\,\langle v \odot u\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1 = c\,(u) + c\,(v) + 1 \cdot \omega^0 \in \mathbb{N}[\omega], \\
& c\,\langle u\#v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1, \\
& c\,\langle u; v\rangle \;=_{\mathrm{def}}\; c\,(u) + c\,(v) + 1, \\
& c\,(u^{\$}) \;=_{\mathrm{def}}\; (c\,(u) + 1) \cdot \omega^1 \in \mathbb{N}[\omega].
\end{aligned}
$$

[$(\_) \cdot \omega^1$ is to account for unknown *iteration count* n in argument $\langle x; n\rangle$ before code expansion.]

Motivation for above **definition**—in particular for this latter iteration case—will become clear with the corresponding case in **proof** of **Descent Lemma** below for *evaluation*

$$ev = ev\,(u, v) \;=_{\mathrm{def}}\; r \,\widehat{\circ}\, \mathrm{wh}\,[\,c_{ev} > 0\,,\, e\,] : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\underline{\perp}} \xrightarrow{r} \mathbb{X}_{\underline{\perp}}$$

defined by a <u>while</u> loop which reads

$$\underline{\text{while}}\ c_{ev}(u) > 0\ \underline{\text{do}}\ (u, x) := e(u, x)\ \underline{\text{od}}.$$

Example: Complexity of *addition* $+ \;=_{\text{by def}}\; s^{\S} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$
\begin{aligned}
& c\,\ulcorner + \urcorner = c\,\ulcorner s^{\S}\urcorner = c\,(\,\ulcorner s\urcorner^{\,\$}) \\
& = (c\,\ulcorner s\urcorner + 1) \cdot \omega^1 = 2 \cdot \omega \in \mathbb{N}[\omega] \quad [\, \equiv\ 0; 2\ \in \mathbb{N}^+\,]
\end{aligned}
$$

Evaluation *step* and *complexity* above are in fact the right ones to give

Basic Descent Lemma: For formally *partially defined* and "nevertheless" *epi-terminating* evaluation map: the defined-arguments PR enumeration of partial map is epi—this by axiom schema (π)—,

$$\begin{aligned}
& ev = ev\,(u,x) \;\; =_{\text{by def}} \;\; r \,\widehat{\circ}\, \mathrm{wh}\,[\, c_{ev} > 0, e\,] : \\
& \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \rightharpoonup \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \xrightarrow{r} \mathbb{X}_{\perp\!\!\!\perp} \\
& (\text{epi-terminating within Theory } \pi\mathbf{R} = \mathbf{PRa} + (\pi))
\end{aligned}$$

i. e. for step $e = e\,(u,x) = (e_{\text{map}}, e_{\text{arg}}) : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp}$ and complexity $c_{ev} = c_{ev}(u,x) \;\; =_{\text{def}} \;\; c\,(u) : \mathrm{PR}\mathbb{X} \to \mathbb{N}[\omega]$, we have Descent *above* $0 \in \mathbb{N}[\omega]$, and Stationarity *at* complexity 0 :

$$\begin{aligned}
\mathbf{PR}\mathbb{X} \vdash\; & c_{ev}(u,x) > 0 \implies c_{ev}\, e\,(u,x) < c_{ev}(u,x) : \\
& \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp\!\!\!\perp} \to \mathbb{N}[\omega] \times \mathbb{N}[\omega] \to \mathbb{2} \;\text{ i. e.} \\
\mathbf{PR}\mathbb{X} \vdash\; & c\,(u) > 0 \implies c\, e_{\text{map}}(u,x) < c\,(u) && \text{(Desc)} \\
& \text{as well as} \\
\mathbf{PR}\mathbb{X} \vdash\; & c\,(u) \doteq 0 \quad [\iff \;\; u \equiv \ulcorner \mathrm{id} \urcorner \;] \\
& \;\; \implies c_{ev}\, e\,(u,x) \doteq 0 \quad [\, \wedge\; e\,(u,x) \doteq (u,x)\,] && \text{(Sta)}
\end{aligned}$$

This with respect to the canonical, *lexicographic,* and—intuitively—*finite-descent* order of polynomial semiring $\mathbb{N}[\omega]$.

Proof: The only non-trivial case $(v,b) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}$ for descent $c_{ev}\, e\,(v,b) < c_{ev}(v,b)$ is iteration case $(v,b) = (u^{\$}, \langle x; n\rangle)$. In this "acute" iteration case we have, first

$$\begin{aligned}
& c\,(u^{[n]}) = c\,(\langle u \odot \langle u \ldots \odot u \rangle \ldots \rangle) \\
& = n \cdot c\,(u) + (n \dot{-} 1)
\end{aligned}$$

proved in detail by induction on n. Whence in fact

$$\begin{aligned} & c_{ev}\, e\,(u^{\$}, \langle x; n \rangle) = c\,(u^{[n]}) \quad \text{(definition of } e) \\ & = n \cdot c\,(u) + (n \dot{-} 1) < (c\,(u) + 1) \cdot \omega \quad (\text{since } \omega > m \text{ for } m \in \mathbb{N}) \\ & \ =_{\text{by def}} \ c\,(u^{\$}) \ =_{\text{by def}} \ c_{ev}(u^{\$}, \langle x; n \rangle). \end{aligned}$$

["+1" in $c\,(u^{\$}) \ =_{\text{def}} \ \omega \cdot c\,(u) + 1$ is to account for the (trivial) case $u^{\$} := \ulcorner \mathrm{id} \urcorner^{\,\$}$ in the above]

Stationarity at complexity $0 \in \mathbb{N}[\omega]$ is obvious **q.e.d.**

Remark: It is important, that complexity—formally—bears only on map codes. Complexity of arguments is subsumed by 'ω' in map term complexity of iterated, within $\mathbb{N}[\omega]$. Just for this end we define complexity c within—Ordinal—$\mathbb{N}[\omega]$. *Basic Descent Lemma* makes plausible **global termination** of $\mathbf{P\widehat{R}a}$ evaluation

$$ev = ev\,(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp}$$

in a suitable framework, here: it **proves** that this (formally) *partial* evaluation map out of $\mathbf{P\widehat{R}a}\mathbb{X}$ *epi-terminates* within Theory $\pi\widehat{\mathbf{R}} = \mathbf{P\widehat{R}a}\mathbb{X} + (\pi)$. This means that evaluation ev has an *onto, epi*—but not retractive—*defined arguments* enumeration

$$\begin{aligned} & d_{ev} = d_{ev}(m, (u, x)) \ =_{\text{by def}} \ (u, x) : \\ & D_{ev} \ =_{\text{by def}} \ \{(m, (u, x)) \,|\, c\ \ell\ e^{m}(u, a) \doteq 0\} \to \mathrm{PR}\mathbb{X} \times \mathbb{X}_{\perp}\,, \end{aligned}$$

this epi-property of CCI ev given by axiom schema (π).

Remark: For **set** theories $\mathbf{T}$ as frame, above *Descent* **shows** that PR map code evaluation ev terminates on all $(u; x) \in \mathrm{PR}\mathbb{X} \times \mathbb{X}$.

In present context, we need an "explicit" Free-Variable Termination **Condition,** in particular for our *basic* evaluation *ev*, and later for its extension ev_d into an evaluation for *argumented deduction trees.*

For a <u>while</u> loop in general—formal **definition** as a *partial* PR map, see chapter 2—of form

$$\mathrm{wh}\,[\,\chi, f\,]\,(a) : A \rightharpoonup A \quad (\text{read: } \underline{\text{while}}\ \chi(a)\ \underline{\text{do}}\ a := f(a)\ \underline{\text{od}}),$$
$$\textbf{define}\ \ [\,m\ \mathit{deff}\ \mathrm{wh}\,[\,\chi, f\,]\,(a)\,]\ =_{\mathrm{def}}\ [\,\neg\,\chi\ f^m(a)\,] : \mathbb{N} \times A \to \mathbb{2}\ :$$

m "defines" argument a for <u>while</u> loop wh $[\,\chi, f\,]$: the loop to *terminate* on this *defined argument* after at most m *steps.*

This gives in addition:

$$[\,m\ \mathit{deff}\ \mathrm{wh}\,[\,\chi, f\,]\,(a)\,]\ \Longrightarrow\ \mathrm{wh}(a) \doteq_A \widehat{\mathrm{wh}}\,(a, m) : A \times \mathbb{N} \to \mathbb{2}\,,$$
$$\widehat{\mathrm{wh}}\,[\,\chi, f\,]\,(a, m)\ =_{\mathrm{by\,def}}\ f^{\S}(a, \min\{n \le m \,|\, \neg\,\chi\ f^n(a)\}) :$$
$$A \times \mathbb{N} \to \mathbb{2},$$

cf. chapter 2 for the details in the general <u>while</u> case: $\widehat{\mathrm{wh}}$ is the (calculation) *rule* of that (formally) partial map.

Things become more elegant for CCI's, because of *stationarity* of CCI's at complexity $0 \in \mathbb{N}[\omega]$:

$$\textbf{PRa} \vdash [\,m\ \mathit{deff}\ \mathrm{wh}\,[\,c > 0, p\,]\,(a)\,]$$
$$= [\,c\ p^m(a) \doteq 0\ \wedge\ \widehat{\mathrm{wh}}(a) \doteq_A p^m(a)\,] :$$
$$A \times \mathbb{N} \to \mathbb{2},\ \text{in particular:}$$
$$\textbf{PRa} \vdash [\,m\ \mathit{deff}\ \ ev\,(u, x)\,]$$
$$= [\,c\,\ell\,e^m(u, x) \doteq 0\ \wedge\ ev\,(u, x) \doteq r\,e^m(u, x)\,] :$$
$$\mathbb{N} \times (\pi\mathrm{R} \times \mathbb{X}_{\underline{\perp}}\,) = \mathbb{N} \times (\mathrm{PRa}\mathbb{X} \times \mathbb{X}_{\underline{\perp}}\,) \to \mathbb{2}.$$

We will use this *given* termination counter "m *deff* ..." only as a (termination) *condition,* in *implications* of form

$$m \ \textit{deff} \ \mathrm{wh}\,[\,c > 0, p\,]\,(a) \implies \chi(a),$$

$\chi = \chi(a)$ a *termination conditioned* predicate. And we will make assertions on formally *partial* maps such as evaluation *ev* and *argumented deduction-tree evaluation* ev_d below mainly in this termination-conditioned, "total" form.

So the main stream of our story takes place in Theories **PRa**, $\pi\mathbf{R} = \mathbf{PRa}+(\pi)$: we go back usually to the **PRa**-building blocks of formally partial maps occuring, in particular to those of *basic evaluation ev* as well as those of *tree evaluation* ev_d to come.

Iteration Domination above gives the following

Dominated Characterisation Theorem for **evaluation:** $ev = ev\,(u, a) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ is characterised by

$$\mathbf{PRa} \vdash \ [\,ev\,(\,\ulcorner \text{bå} \urcorner\,, x) \doteq \text{bå}(x)\,]$$

as well as, within **PRa** :

$$[\,m \ \textit{deff} \ \ ev\,(v \odot u, x)\,] \implies$$
$$ev\,(\langle v \odot u\rangle, x) \doteq ev\,(v, ev\,(u, x)),$$

and

$$[\,m \ \textit{deff} \ \ ev\,(\langle u\#v\rangle, \langle x; y\rangle)\,] \implies$$
$$ev\,(\langle u\#v\rangle, \langle x; y\rangle) \doteq \langle ev\,(u, x); ev\,(v, y)\rangle,$$
$$[\,m \ \textit{deff} \ \ ev\,(\langle u; v\rangle, z)\,] \implies$$
$$ev\,(\langle u; v\rangle, z) \doteq \langle ev\,(u, z); ev\,(v, z)\rangle,$$

as well as

$$\begin{aligned}
& ev\,(u^{\$}, \langle x; \ulcorner 0 \urcorner \rangle) \doteq x, \\
& [\, m \ \mathit{deff} \ \ ev\,(u^{\$}, \langle x; \nu(s\,n) \rangle \,] \implies : \\
& [\, m \ \mathit{deff} \ \text{all} \ \ ev \ \text{below}\,] \\
& \wedge \ ev\,(u^{\$}, \langle x; \nu(s\,n) \rangle) \doteq ev\,(u, ev\,(u^{\$}, \langle x; \nu\, n \rangle)).
\end{aligned}$$

Remark: Evaluation here is evaluation of map codes internally *composed* with their *arguments*—suggestion of Joseph Helfer:

$$ev(u, x) = ev(u \odot x, \ulcorner 0 \urcorner) = ev(u \ulcorner \circ \urcorner x, \ulcorner 0 \urcorner) : \mathrm{PRa}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X},$$

in particular

$$\begin{aligned}
& ev(\ulcorner f \urcorner, \mathrm{num}(\underline{n})) \check{\doteq} ev(\ulcorner f \urcorner \odot \ulcorner s \urcorner \odot \ldots \odot \ulcorner s \urcorner \odot \ulcorner 0 \urcorner, \ulcorner 0 \urcorner) \\
& \check{\doteq} ev(\ulcorner f \circ s \circ \ldots \circ s \circ 0 \urcorner, \ulcorner 0 \urcorner) : \\
& \mathbb{N} \rightharpoonup \mathbb{X}, \ f \ \text{meta-free in } \mathbf{PRa}(\mathbb{N}, \mathbb{X}).
\end{aligned}$$

Proof of **Theorem** by Primitive Recursion (Peano Induction) on $m \in \mathbb{N}$ free, via case distinction on codes w, and arguments $z \in \mathbb{X}$ appearing in the different cases of the asserted conjunction (case w one of the basic map constants being trivial). All of the following—**induction step**—is situated in **PRa**, read: $\mathbf{PRa} \vdash$ etc. If you are interested first in the negative results for **set** theories **T**, you can read it "$\mathbf{T} \vdash \ \ldots$" but **T** still deriving properties just of $\mathbf{PR}\mathbb{X}$ *map codes.*

- case $(w, z) = (\langle v \odot u \rangle, x)$ of an (internally) *composed,* subcase $u = \ulcorner \mathrm{id} \urcorner$: obvious.

Non-trivial subcase $(w, z) = (\langle v \odot u \rangle, x)$, $u \neq \ulcorner \mathrm{id} \urcorner$:

$$\begin{aligned}
& m+1 \ \mathit{deff} \ \ ev\,(\langle v \odot u \rangle, x) \implies : \\
& ev\,(\langle v \odot u \rangle, x) \doteq e^{\S}((\langle v \odot e_{\mathrm{map}}(u, x) \rangle, e_{\mathrm{arg}}(u, x)), m) \\
& \qquad\qquad \text{by iterative definition of } ev \text{ in this case} \\
& \doteq ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \\
& \qquad\qquad \text{by induction hypothesis on } m \\
& \implies : \\
& m+1 \ \mathit{deff} \ \ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \\
& \wedge \ ev\,(v, ev\,(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x))) \doteq ev\,(v, ev\,(u, x)) \ :
\end{aligned}$$

The latter implication "holds" same way back, by the same induction hypothesis on m (map code v unchanged) in both directions of PR reasoning.

- case $(w, z) = (\langle u \# v \rangle, \langle x; y \rangle)$ of an (internal) *Cartesian product:* Obvious by definition of ev on a Cartesian product map codes. Pay attention to arguments out of $\mathbb{X} \smallsetminus \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle$ evaluated into $\underline{\perp}$ in this case (and in similar cases). In more detail:

$$\begin{aligned}
& ev\,(w, z) := \\
& ev\,(\langle u \# v \rangle, \langle x; y \rangle) \\
& \quad =_{\text{by def}} \ ev\,(\langle e_{\mathrm{map}}(u, x) \# e_{\mathrm{map}}(v, y) \rangle, \langle e_{\mathrm{arg}}(u, x), e_{\mathrm{arg}}(v, y) \rangle) \\
& \qquad \doteq \langle ev(e_{\mathrm{map}}(u, x), e_{\mathrm{arg}}(u, x)), ev(e_{\mathrm{map}}(v, y), e_{\mathrm{arg}}(v, y)) \rangle \\
& \qquad\quad \in \langle \mathbb{X} \dot{\times} \mathbb{X} \rangle
\end{aligned}$$

- alternatively (or both): case $(w, z) = (\langle u; v \rangle, z)$ of an internal induced:

$$ev(w, z) \doteq \langle ev(u, z), ev(v, z)\rangle \in \langle \mathbb{X} \dot{\times} \mathbb{X}\rangle.$$

- case $(w, z) = (u^{\$}, \langle x; \ulcorner 0 \urcorner\rangle)$ of a null-fold (internally) iterated: again obvious.
- case $(w, z) = (u^{\$}, \langle x; \nu(s\,n)\rangle)$ of a genuine (internally) iterated:

$$\begin{aligned}
&m + 1 \textit{ deff } ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \implies \\
&m + 1 \textit{ deff } \text{all instances of } ev \text{ below, and:} \\
&ev\,(u^{\$}, \langle x; \nu(s\,n)\rangle) \\
&\doteq ev\,(e_{\mathrm{map}}(u^{\$}, \langle x; \nu(s\,n)\rangle), e_{\mathrm{arg}}(u^{\$}, \langle x; \nu(s\,n)\rangle)) \\
&\doteq ev\,(u^{[n+1]}, x) \doteq ev\,(\langle u \odot u^{[n]}\rangle, x) \doteq ev\,(u, ev\,(u^{[n]}, x)) \\
&\qquad \text{the latter by induction hypothesis on } m, \\
&\qquad \text{case of internal composed} \\
&\doteq ev\,(u, \langle ev\,(u^{\$}, x); \nu\,n\rangle) : \text{same way back.}
\end{aligned}$$

This shows the (remaining) predicative *iteration* equations "anchor" and "step" for an (internally) iterated $u^{\$}$, and so **proves** fullfillment of the above **Double Recursive** system of equations for $ev : \mathrm{PRa}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ subordinated to *global* evaluation $ev : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$ **q.e.d.**

Characterisation Corollary: Evaluation—$\mathbf{P\widehat{R}a}$ map—

$$ev = ev\,(u, x) : \mathrm{PR}\mathbb{X} \times \mathbb{X} \rightharpoonup \mathbb{X}$$

defined as *Complexity Controlled Iteration*—CCI—with complexity values in Ordinal $\mathbb{N}[\omega]$, epi-terminates in Theory $\pi\widehat{\mathbf{R}}$, by definition of this theory strengthening $\widehat{\mathbf{PRa}}$, and satisfies there the characteristic Double-Recursive equations above for evaluation ev. It *terminates,* when situated in a **set** theory $\mathbf{T}$ (with all properties), since there complexity receiving Ordinal $\mathbb{N}[\omega]$ has (only) finite Descent, in terms of existential quantification.

Evaluation Family:

- *evaluation* ev "splits" into transformation

$$[\ ev_{A,B} = ev_{A,B}(u,a) : \lceil A, B \rceil \times A \rightharpoonup B\ ]_{A,B \in \mathbf{PRa}\mathbb{X}}$$

  with all of the above characteristic properties "split".

- If you combine—within Theory **PRa**—above evaluation <u>family</u> with the isomorphisms $\nu_A : A \xrightarrow{\cong} \dot{A}$ and $\nu_B^{-1} : \dot{B} \xrightarrow{\cong} B,\ A, B$ within **PRa**, you get evaluation

$$\begin{aligned}&[\ ev_{A,B} = ev_{A,B}(u,a) : \\ &\lceil A, B \rceil_{\mathbf{PRa}} \times A \xrightarrow{\cong} \lceil \dot{A}, \dot{B} \rceil_{\mathbf{PRa}\mathbb{X}} \times \dot{A} \rightharpoonup \dot{B} \xrightarrow{\cong} B\ ]_{A,B \in \mathbf{PRa}}\end{aligned}$$

  which has—mutatis mutandis—obviously all properties of evaluation above.

Comment: Universal set $\mathbb{X} \subset \mathbb{N}$ seems to give a good service: without it, we would have be forced (?) to define evaluation ev as a <u>family</u>

$$ev = [\, ev_{A,B} : \lceil A, B \rceil \times A \rightharpoonup B\,]_{A,B}$$

<u>meta-indexed</u> over pairs of Objects of Theory **PRa**, as is usual in Category Theory for *axiomatically* given evaluation

$$\epsilon = [\, \epsilon_{A,B} : B^A \times A \to B\,]_{A,B \in \mathbf{Obj_C}}$$

C a (Cartesian) Closed Category in the sense of EILENBERG & KELLY 1966 and LAMBEK & SCOTT 1986. (Observe our typographic distinction between the two "evaluations").

At least formally, a *constructive* **definition** of evaluation as one single—formally partial—**PRa** map $ev = ev\,(u, x) : \lceil \mathbb{X}, \mathbb{X} \rceil \times \mathbb{X} \rightharpoonup \mathbb{X}$ is "necessary" or at least makes things simpler.

So both, the typified approach—traditional in Categorical main stream, as well as the EHRESMANN type one starting with just one *class* of maps—and partially defined composition—are usefull in our context: *Universal set* $\mathbb{X}$—of *internal numerals* $\nu(n)$ and (nested) pairs thereof—makes the join.

Evaluation Objectivity: We "rediscover" here the logic *join* between the *Object Language* level and the external PR Metamathematical level, join by externalisation via evaluation ev above. The corresponding, very plausible Theorem says that evaluation ev *mirrors* "concrete" *codes,* $\ulcorner f \urcorner$ of maps $f : A \to B$ of Theory **PRa**.

Objectivity Theorem: Evaluation ev is *objective,* i. e. for each *single,* (meta <u>free</u>) $f : A \to B$ in Theory **PRa** itself, we have

$$\mathbf{PRa} \vdash \; ev_{A,B}(\ulcorner f \urcorner, a) = f(a) : A \to B, \text{ symbolically:}$$
$$\mathbf{PRa} \vdash \; ev_{A,B}(\ulcorner f \urcorner, \_\,) = f : A \to B.$$

A fortiori: $\pi\mathbf{R} \vdash \; ev_{A,B}(\ulcorner f \urcorner, a) = f(a) : \mathbb{X} \sqsupset A \to B \sqsubset \mathbb{X}.$

Here formally partial *code evaluation*
$ev_{A,B} = ev_{A,B}(u, a) : \lceil A, B \rceil \times A \rightharpoonup B$
of theories **PRa** as well as $\pi\mathbf{R}$ is given as

$$\lceil A, B \rceil \times A \xrightarrow{\sqsubset \times \sqsubset} \mathbf{PR}\mathbb{X} \times \mathbb{X} \xrightarrow{ev} \mathbb{X} \xrightarrow{\mathrm{re}_B} B,$$

$\mathrm{re}_B : \mathbb{X} \twoheadrightarrow B$ being the canonical retraction to (canonical) injection $B \overset{\sqsubset}{\to} \mathbb{X}$, B assumed pointed, by $b_0 : \mathbb{1} \to B$ say.

Remark: For such f fixed,

$$ev\,(\ulcorner f \urcorner, a) = ev\,\widehat{\circ}\,(\ulcorner f \urcorner, a) : A \to \lceil A, B \rceil \times A \rightharpoonup B$$

is in fact a **PRa** map $ev\,(\ulcorner f \urcorner, a) : A \to B$, although in **proof** of the **Theorem** intermediate steps are formally $\mathbf{P\widehat{R}a}$ equations " $\widehat{=}$ ". But $\mathbf{PRa} \sqsubset \mathbf{P\widehat{R}a}$ is a diagonal monoidal PR Embedding.

Proof of *Evaluation Objectivity* by **first:** External structural recursion on the nesting depth depth $[\,f\,]$ ("bracket depth") of **PRa**-map $f : A \to B$ in question, seen as external code $f \in \underline{\mathbb{N}}$, and **second:** in case of an iterated, $g^\S = g^\S(a, n) : A \times \mathbb{N} \to A$, by PR on *iteration count* $n \in \mathbb{N}$. This uses *Double Recursive Characterisation* of evaluation ev.

Otherwise the **proof** is parallel to that of Theorem above, without the need of explicit domination **q.e.d.**

Evaluation Remark: We saw that *Coding* $\langle f : A \to B \rangle \mapsto \langle \ulcorner f \urcorner : \mathbb{1} \to \lceil A, B \rceil \rangle$ admits evaluation as (meta) retraction. But are **PRa** and $\mathrm{PRa}^{\mathbb{X}} / \dot{\asymp}$ (meta) isomorphic as categories?

On Category Conference at Louvain-la-Neuve 1988 A. Pitts answered to this question by a categorical "No".

In fact, there is a level conflict: evaluation can not decide, if $u \in \lceil A, B \rceil$ is a concrete code of form $\ulcorner f \urcorner$. These concrete codes are only meta enumerated.

But things to come will demonstrate that evaluation externalises/objectivates the whole categorical structure of $\mathrm{PRa}\mathbb{X}$ including in particular substitution, Cartesian product, and iteration of (variable) map codes, and that it objectivates as well internal PR equality

$\check{=}$ into predicative (objective) equality $\dot{=}$, all of this under (plausible) condition the evaluations to terminate: *Termination Conditioned Soundness* to come.

Remark on PR map code evaluation within set theory frame: There this evaluation always terminates, in terms of formal *existence* of an $m \in \mathbb{N}$ which annihilates complexity $c\,e^m(u,a) \in \mathbb{N}[\omega]$. This since complexity decreases above 0 with increasing m, and since (Ordinal) $\mathbb{N}[\omega]$ admits only finite descending chains.

All of the other properties of this evaluation carry over to **sets** as frame, since **sets** is an extension of frames $\mathbf{PRa}, \pi\mathbf{R}, \mathbf{PRa}\mathbb{X}, \mathbf{P}\widehat{\mathbf{R}}\mathbf{a}, \pi\widehat{\mathbf{R}}$ considered above (and below).

Chapter 5

Consistency and Inconsistency

Following a suggestion of Joseph, I discuss *Consistency Provability* via *Soundness* of PR-evaluation in this chapter simultaneously for frame theory $\mathbf{S}$ say, $\mathbf{S}$ taken PR Descent Theory $\pi\mathbf{R}$ or an extension of $\pi\mathbf{R}$, in particular $\mathbf{S}$ a **set** theory $\mathbf{T}$.

For "both" $\mathbf{S} = \mathbf{T}$ and $\mathbf{S} = \pi\mathbf{R}$ we will obtain (free-variable) Consistency formula $\mathrm{Con}_{\mathbf{S}}$ as an $\mathbf{S}$ Theorem, not expected, since it contradicts in case $\mathbf{S} = \mathbf{T}$ quantified Arithmetic Gödel's 2nd *Incompleteness Theorem,* Gödel ***assuming*** ω-consistency for such $\mathbf{T}$.

For Objectivation of internal $\mathbf{PR}\mathbb{X}$ equality $u \doteq v$ into $ev(u, x) \doteq ev(v, x)$ within constructive frame Theory $\pi\mathbf{R}$, namely *PR Descent Theory* $\pi\mathbf{R} = \mathbf{PRa} + (\pi)$ with axiom schema (π) of non-infinite complexity descent of Complexity Controlled Iterations (complexity values in Ordinal $\mathbb{N}[\omega]$), we have to control termination of ev along the map codes of whole deduction trees dtree_k and their spread down arguments: *Deduction-tree evaluation* ev_d, iteration of evaluation step e_d controlled by descending complexity c_d. We do this by recursive case distinction on internalised **axiom** schema dedu_k at hand, top down argumented with x, x_i, x_j etc. suitable, starting with $x \in \mathbb{X}$ suitable.

This Objectivation works for any theory **S** extending PR descent Theory $\pi\mathbf{R}$, in particular for ($\pi\mathbf{R}$ itself and) **set** theories **T**. In first reading—*Inconsistency*—take **set** theory **T** as frame.

General argumented $\mathbf{PR}\mathbb{X}$ deduction tree has form

$$\mathrm{dtree}_k/x \quad = \quad \frac{u_k/x \qquad v_k/x}{\dfrac{u_i/x_i \qquad v_i/x_i}{\mathrm{dtree}_{ii}/x_{ii}\ \mathrm{dtree}_{ji}/x_{ji}} \qquad \dfrac{u_j/x_j \qquad v_j/x_j}{\mathrm{dtree}_{ij}/x_{ij}\ \mathrm{dtree}_{jj}/x_{jj}}}$$

x_i, x_j etc. the arguments spread down the tree from top argument $x \in \mathbb{X}$: constants or free on $\mathbb{X}$, or more general $\mathbb{X}$ terms. Some branches may be empty, if so are all of them, then $\mathrm{depth}(\mathrm{dtree}_k/x) = \mathrm{depth}(\mathrm{dtree}_k) = 0$, these trees are *flat.* Moreover, some non-empty branches may be not (yet) provided with arguments.

Argumented tree evaluation step e_d on such a tree is **defined** node-wise recursively—PR—in everyday cases via basic map code evaluation step e as

$$e_d(\mathrm{dtree}_k/x) \quad =$$

$$\frac{e(u_k, x) \quad e(v_k, x)}{\dfrac{e(u_i, x_i) \quad e(v_i, x_i)}{e_d(\mathrm{dtree}_{ii}/x_{ii}) \ \ e_d(\mathrm{dtree}_{ji}/x_{ji})} \qquad \dfrac{e(u_j, x_j) \quad e(v_j, x_j)}{e_d(\mathrm{dtree}_{ij}/x_{ij}) \ \ e_d(\mathrm{dtree}_{jj}/x_{jj})}}$$

Deviations are treated explicitely in proof of Soundness Theorem below.

Evaluation complexity of such a tree is **defined** as the *sum* of all map code complexities in the nodes of the (argumented) tree, sum taken within polynomial Ordinal $\mathbb{N}[\omega]$.

It descends strictly with iterative application of step e_d, until all of these map codes in the tree vanish or become $\ulcorner \mathrm{id} \urcorner : c_d \doteq 0$.

Iterative evaluation result then is

$$ev_d(\mathrm{dtree}_k/x) = \langle \ulcorner \mathrm{id} \urcorner / ev(u_k, x); \ulcorner \mathrm{id} \urcorner / ev(v_k, x) \rangle$$

with $ev(u_k, x) \doteq ev(v_k, x)$, as will be shown in **proof** of *ES Theorem on Termination Conditioned Soundness* below.

Some of these evaluation steps will be displayed explicitely within that proof, including recursive arguments propagation from top to bottom.

5.1 Termination Conditioned Soundness

We can now—in particular in parallel to forgoing chapter—develop all ingredients needed in **proof** of

ES[1] **Theorem on Termination-Conditioned Soundness:**

For PR Theory **PRa** [2] and internal notion of equality $\check{\doteq} = \check{\doteq}_k : \mathbb{N} \to \mathrm{PR}\mathbb{X} \times \mathrm{PR}\mathbb{X}$, dtree_k the k th deduction tree of Universe Theory $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$, we have:

(i) *Termination-Conditioned **Inner** Soundness:*

With $r = r(u, x) = x : \mathrm{PR}\mathbb{X} \times \mathbb{X} \to \mathbb{X}$ right projection:

$$\begin{aligned} \mathbf{PRa} \vdash\ & \langle u \check{\doteq}_k v \rangle \doteq \mathrm{root}\ (\mathrm{dtree}_k) \\ & \wedge\ m\ \mathit{deff}\ \ ev_d\,(\mathrm{dtree}_k/x) \\ & \Longrightarrow\ ev\,(u, x) \doteq ev\,(v, x)\,. \qquad (\bullet) \end{aligned}$$

explicitely:

$$\begin{aligned} \mathbf{PRa} \vdash\ & u \check{\doteq}_k v\ \wedge\ c_d\, e_d^m\,(\mathrm{dtree}_k/x) \doteq 0 \\ \Longrightarrow\ & ev\,(u, x) \doteq e^m(u, x) \doteq e^m(v, x) \\ & \doteq ev\,(v, x), \qquad (\bullet) \end{aligned}$$

free map-code Variables u, v, variable x free in Universal set $\mathbb{X}$.

[*Argumentation* dtree_k/x of dtree_k and definition of *argumented tree evaluation* ev_d based on its evaluation step e_d and complexity c_d is by merged recursion on depth(dtree_k), within **proof** below]

In words, this "m-Truncated", "m-Dominated" Inner Soundness says that Theory **PRa** derives:

---

[1] *Evaluation Soundness*

[2] presumably *not* directly for $\pi\mathbf{R}$ with respect to its own internal equality, without assumption of "π-consistency," in this regard RCF 2 contains an error

If *for an internal* **PR**$\mathbb{X}$ equation $u \check{\doteq}_k v$ argumented deduction tree dtree_k/x *for* $u \check{\doteq}_k v$, argumented *with* $x \in \mathbb{X}$, *admits complete* argumented-tree evaluation, i. e.

if *tree-evaluation becomes* ***completed*** *after a finite number* m *of evaluation steps,*

then *both sides of this* internal (!) equation *are completely* ***evaluated*** *on* x *by (at most)* m *steps* e *of* basic *evaluation* ev, *into* ***equal values.***

Substituting in the above "concrete" codes into u resp. v, we get, by *Objectivity* of evaluation ev, formally "mutatis mutandis":

(ii) *Termination-Conditioned Objective Soundness for Map Equality:*

For **PRa** maps $f, g : \mathbb{A} \to B$:

$$\begin{aligned}\mathbf{PRa} \vdash [\, \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \ \wedge\ m \ \mathit{deff}\ ev_d(\mathrm{dtree}_k/a)\,] \\ \implies f(a) \doteq_B r\, e^m(\ulcorner g \urcorner, a) \doteq_B g(a),\ a \in A \text{ free :}\end{aligned}$$

If *an internal PR deduction-tree for (internal) equality of* $\ulcorner f \urcorner$ *and* $\ulcorner g \urcorner$ *is available, and* ***if*** *on this tree—top down argumented with a in A—tree evaluation* ***terminates, then*** *equality* $f(a) \doteq_B g(a)$ *of* f *and* g *at this argument is the consequence.*

(iii) Specialising this to case of $f := \chi : A \to \mathbb{2}$ a PR *predicate* and to $g := \mathrm{true}_A : A \to \mathbb{2}$ we eventually get

Termination-Conditioned Objective Logical Soundness:

$$\mathbf{PRa} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \chi \urcorner) \ \wedge\ m\ \mathit{deff}\ ev_d(\mathrm{dtree}_k/a) \implies \chi(a) :$$

> ***If*** *tree-evaluation of an internal* ***deduction*** *tree for a free variable PR predicate* $\chi : A \to \mathbb{2}$*—the tree* argumented *with* $a \in A$*—* ***terminates*** *after a finite number* m *of evaluation steps,* ***then*** $\chi(a) \doteq \mathrm{true}$ *is the consequence,* within **PRa** as well within its extensions $\pi\mathbf{R}$ and **set** theory.

Remark:

• On replacement in the above Theorem—and **proof** below—**PRa** as frame by theories **S** extending **PRa**, theorem and proof are inherited.

• For frame taken Quantified Arithmetic—**set** theory **T**—everything works even without the *termination conditioning* clauses m *deff* etc. :

All evaluations terminate, you may replace each instance of clause m *deff* etc. by the theorem $\exists\, m\ [\, m$ *deff* etc.$\,]$ and get this way rid of all clauses m *deff* etc. "But" Objectivation of $\check{\doteq}$ into $\doteq$ is then only into theory **T**, and we get as **Corollary** for **set** theories **T** :

T-Framed Soundness of Theory PRa :

$$\begin{aligned}
&\mathbf{T} \vdash\ \exists\, k\ u \check{\doteq}_k v \implies \forall\, x\, ev(u,x) \doteq ev(v,x), \\
&\mathbf{T} \vdash\ \exists\, k\ \ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \implies \forall\, a\ f(a) \doteq g(a), \\
&\mathbf{T} \vdash\ \exists\, k\ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \chi \urcorner\,) \implies \forall\, a\, \chi(a).
\end{aligned}$$

Remark to proof below: in present case of frame theory **PRa** (and stronger Theory $\pi\mathbf{R}$) we have to *control* all evaluation step iterations, and we do that by control of iterative evaluation ev_d of whole argumented deduction trees, whose recursive **definition** will be—merged—part of this proof.

Proof of—basic—*Termination-Conditioned **Inner** Soundness,* i. e. of implication (•) in *ES Theorem* is by induction on deduction tree counting index $k \in \mathbb{N}$ counting family $\mathrm{dtree}_k : \mathbb{N} \to \mathrm{Bintree}$, starting with (flat) $\mathrm{dtree}_0 = \langle \ulcorner \mathrm{id} \urcorner \check{\doteq}_0 \ulcorner \mathrm{id} \urcorner \rangle$. $m \in \mathbb{N}$ is to dominate argumented-deduction-tree evaluation ev_d to be recursively defined below: *condition*

m *deff* $ev_d(\mathrm{dtree}_k/x)$, step e_d, complexity c_d.

We argue by *recursive case distinction* on the form of the top up-to-two layers—top (implicational) deduction—dedu_k/x of argumented deduction tree dtree_k/x at hand.

Flat **SuperCase** $\mathrm{depth}(dtree_k) = 0$, i. e. SuperCase of *unconditioned,* axiomatic (internal) *equation* $u \check{\doteq}_k v$:

The first involved of these cases is *associativity* of (internal) *composition:*

$$\mathrm{dtree}_k \ = \ \langle\langle w \odot v\rangle \odot u\rangle \check{\doteq}_k \langle w \odot \langle v \odot u\rangle\rangle$$

In this case—no need of a recursion on k—

$$\begin{aligned}\mathbf{PRa} \vdash m \textit{ deff } & ev_d(\mathrm{dtree}_k/x) \implies \\ & [\, m \textit{ deff } ev\,(\langle w \odot v\rangle \odot u, x)\,] \\ & \wedge [\, m \textit{ deff } ev(\langle w \odot v\rangle, ev\,(u,x)) \\ & \wedge [\, m \textit{ deff } ev(w, ev(v, ev(u,x))) \\ & \wedge [\, m \textit{ deff } ev(w, ev(\langle v \odot u\rangle, x)) \\ & \wedge [\, m \textit{ deff } ev\,(\langle w \odot \langle v \odot u\rangle\rangle, x)\,] \wedge\end{aligned}$$

$$\begin{aligned}& ev\,(\langle w \odot v\rangle \odot u, x) \doteq ev\,(\langle w \odot v\rangle, ev\,(u,x)) \\ & \doteq ev\,(w, ev\,(v, ev\,(u,x))) \\ & \doteq ev\,(w, ev\,(\langle v \odot u\rangle, x)) \doteq ev\,(w \odot \langle v \odot u\rangle, x).\end{aligned}$$

This proves assertion $(\bullet)$ in present *associativity-of-composition* case. [New in comparison to previous *Inconsistency* chapter is here only the "preamble" m *deff* etc.]

Analogous **Proof** for the other **flat,** equational cases, namely *Reflexivity of Equality, Left and Right Neutrality* of $\mathrm{id} \ =_{\text{by def}} \ \mathrm{id}_{\mathbb{X}}$, all substitution equations for the map constants, Godement's equations for the induced map as well as Fourman's *uniqueness equation (!) for the induced map.*

Godement's equations $\ell \circ (f, g) = f, \ r \circ (f, g) = g$:

$$
\begin{aligned}
&m \ \mathit{deff} \ ev \text{ etc.} \implies \\
&ev(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z) \doteq r\, e^m(\ulcorner \mathring{\ell} \urcorner \odot \langle u; v\rangle, z) \\
&\doteq \mathring{\ell}(\langle ev(u, z); ev(v, z)\rangle) \doteq ev(u, z),
\end{aligned}
$$

analogously for composition with right projection.

Fourman's equation $(\ell \circ h, r \circ h) = h$:

$$
\begin{aligned}
&m \ \mathit{deff} \ ev \text{ etc.} \implies \\
&ev(\langle \ulcorner \mathring{\ell} \urcorner \odot w; \ulcorner \mathring{r} \urcorner \odot w\rangle, z) \\
&\doteq \langle ev(\ulcorner \mathring{\ell} \urcorner, ev(w, z)); ev(\ulcorner \mathring{r} \urcorner, ev(w, z))\rangle \\
&\doteq \langle \mathring{\ell}(ev(w, z)); \mathring{r}(ev(w, z))\rangle \doteq ev(w, z)
\end{aligned}
$$

by Fourman's equation on Objective level.

Now here are the **proofs**—with preambles—of $(\bullet)$, for the last equational case, the

Iteration **step,** case of *genuine iteration equation*

$$\text{dtree}_k \;=\; \langle u^{\$} \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle \check{\doteq}_k u \odot u^{\$} \rangle :$$

$$
\begin{aligned}
\mathbf{PRa} \vdash\ & m\ \textit{deff}\ \ ev_d(\text{dtree}_k/\langle y; \nu(n)\rangle) \implies \\
& m\ \textit{deff}\ \ \textit{all instances of ev below, and:} \\
& ev\,(u^{\$} \odot \langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle) && (1) \\
& \doteq ev\,(u^{\$}, ev(\langle \ulcorner \text{id} \urcorner \# \ulcorner s \urcorner \rangle, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u^{\$}, \langle y; \nu(s\,n)\rangle) \\
& \doteq ev\,(u^{[s\,n]}, y) \quad \text{(by definition of } ev \text{ step } e) \\
& \doteq ev\,(u \odot u^{[n]}, y) \\
& \doteq ev\,(u, ev(u^{\$}, \langle y; \nu(n)\rangle)) \\
& \doteq ev\,(u \odot u^{\$}, \langle y; \nu(n)\rangle). && (2)
\end{aligned}
$$

Proof of Termination-Conditioned Inner Soundness for the remaining *deep*—genuine HORN **cases**—for dtree_k , HORN type (at least) at *deduction* of *root:*

Transitivity-of-equality case: with map code variables u, v, w we start here with argument-free deduction tree

$$\text{dtree}_k \quad = \quad \cfrac{u \,\check{\doteq}_k\, w}{\cfrac{u \,\check{\doteq}_i\, v}{\text{dtree}_{ii} \quad \text{dtree}_{ji}} \qquad \cfrac{v \,\check{\doteq}_j\, w}{\text{dtree}_{ij} \quad \text{dtree}_{jj}}}$$

It is argumented with argument x say, recursively spread down:

$$\text{dtree}_k/x \quad = \quad \dfrac{u/x \quad w/x}{\dfrac{u/x \quad v/x}{\text{dtree}_{ii}/x_{ii} \quad \text{dtree}_{ji}/x_{ji}} \qquad \dfrac{v/x \quad w/x}{\text{dtree}_{ij}/x_{ij} \quad \text{dtree}_{jj}/x_{jj}}}$$

Spreading down arguments from upper level down to 2nd level must/is given explicitely, further arguments spread down is then recursive by the type of deduction (sub)trees $\text{dtree}_i, \text{dtree}_j,\ i, j < k$.

Now by induction hypothesis on i, k we have for tree evaluation ev_d :

$$\begin{aligned}
&m \textit{ deff } ev_d(\text{dtree}_k/x) \\
&\ \Longrightarrow m \textit{ deff } ev_d(\text{dtree}_i/x), ev_d(\text{dtree}_j/x) \wedge \\
&ev_d(\text{dtree}_i/x) \doteq \langle \ulcorner \text{id} \urcorner / ev(u,x) \doteq \ulcorner \text{id} \urcorner / ev(v,x) \rangle \\
&\wedge\ ev_d(\text{dtree}_j/x) \doteq \langle \ulcorner \text{id} \urcorner / ev(v,x) \doteq \ulcorner \text{id} \urcorner / ev(w,x) \rangle \\
&\ \Longrightarrow ev(u,x) \doteq ev(v,x) \wedge ev(v,x) \doteq ev(w,x) \\
&\ \Longrightarrow ev(u,x) \doteq ev(w,x).
\end{aligned}$$

and this is what we wanted to show in present transitivity of equality case.

[Transitivity **axiom** for equality is a main reason for necessity to consider (argumented) deduction trees: intermediate map code equalities '$\check{=}$' in a transitivity chain must be each evaluated, and pertaining deduction trees may be of arbitrary high evaluation complexity]

Case of **symmetry** axiom schema for equality is now obvious.

Compatibility Case of composition with equality[3]

$$\mathrm{dtree}_k/x \quad = \quad \cfrac{\langle v \odot u\rangle/x \,\check{\doteq}_k\, \langle v \odot u'\rangle/x}{\cfrac{u/x \,\check{\doteq}_j\, u'/x}{\mathrm{dtree}_{ij}/x \quad \mathrm{dtree}_{jj}/x}}$$

By induction hypothesis on $j < k$

$$\begin{aligned}
&m \ \mathit{deff} \ ev_d(\mathrm{dtree}_k/x) \implies \\
&m \ \mathit{deff} \ ev_d(\mathrm{dtree}_j/x) \implies \\
&ev(u,x) \doteq ev(u',x) \implies \\
&ev(v \odot u, x) \doteq ev(v, ev(u,x)) \doteq ev(v, ev(u',x)) \\
&\doteq ev(v \odot u', x)
\end{aligned}$$

by dominated characterisic equations for *ev* and Leibniz' substitutivity, q.e.d. in this 1st compatibility case.

Spread down arguments is more involved in

Case of composition with equality in second composition factor: argument spread down merged with tree evaluation ev_d and proof of result.

[3] this simplified version has been suggested by Joseph

$$\mathrm{dtree}_k/x \quad = \quad \cfrac{\dfrac{\langle v \odot u\rangle/x \qquad \langle v' \odot u\rangle/x}{v \,\check{\doteq}_i\, v'}}{\mathrm{dtree}_{ii} \qquad \mathrm{dtree}_{ji}}$$

[Here dtree_i is not (yet) provided with argument, it *is* argumented during top down tree evaluation below]

$$\begin{aligned}
&m\ \mathit{deff}\ \ ev_d(\mathrm{dtree}_k/x) \implies \\
&m\ \mathit{deff}\ \text{ all instances of } ev \text{ below, and:} \\
&ev(\langle v \odot u\rangle, x) \doteq ev(v, ev(u,x)) \doteq ev(v', ev(u,x)) \qquad\qquad (*) \\
&\ \doteq ev(\langle v' \odot u\rangle, x).
\end{aligned}$$

$(*)$ holds by Leibniz' substitutivity and

$$\begin{aligned}
&m\ \mathit{deff}\ \ ev_d(\mathrm{dtree}_k/x) \implies \\
&m\ \mathit{deff}\ \ ev_d(\mathrm{dtree}_i/ev(u,x)) \\
&[\,\textit{argumentation of } \mathrm{dtree}_i \textit{ with} \\
&ev(u,x)\text{—}\textit{calculated en cours de route}, \\
&\text{extra } \textbf{definition} \text{ of } e_d\,] \\
&\ \implies \\
&m\ \mathit{deff}\ \ ev(v, ev(u,x)) \doteq ev(v', ev(u,x)),
\end{aligned}$$

by induction hypothesis on $i < k$: The hypothesis is independent of substituted argument, provided—and this is here the case—that dtree_i is evaluated on that argument, in $m' < m$ steps, m' suitable (minimal).

This proves assertion (•) in this 2nd compatibility case.

Case of **compatibility** of Cartesian procuct of maps with equality is analogous to compatibilities above, even easier, since the two map codes concerned are completely independent from each other.

Case of **compatibility** of forming the **induced map** with **equality** follows from the above combined with compatibility of composition with equality. It can also be obtained directly straight forward.

(Final) Case of Freyd's (internal) **uniqueness** of the *initialised iterated,* is **case**

$$\mathrm{dedu}_k/\langle y;\nu(n)\rangle \quad = \quad \frac{w/\langle y;\nu(n)\rangle \,\check{\doteq}_k\, \langle v^{\$} \odot \langle u\# \ulcorner\mathrm{id}\urcorner \rangle/\langle y;\nu(n)\rangle\rangle}{\mathrm{root}\,(t_i) \qquad\qquad\qquad\qquad \mathrm{root}\,(t_j)}$$

where

$$\mathrm{root}\,(t_i) = \langle w \odot \langle \ulcorner\mathrm{id}\urcorner\,;\, \ulcorner 0\urcorner \odot \ulcorner\Pi\urcorner \rangle/y \,\check{\doteq}_i\, u/y\rangle,$$
$$\mathrm{root}\,(t_j) = \langle w \odot \langle \ulcorner\mathrm{id}\urcorner \# \ulcorner s\urcorner \rangle/\langle y;\nu(n)\rangle \,\check{\doteq}_j\, \langle v \odot w\rangle/\langle y;\nu(n)\rangle\rangle$$

Comment: w is here an internal *comparison candidate* fullfilling the same internal PR equations as $\langle v^{\$} \odot \langle u\# \ulcorner\mathrm{id}\urcorner \rangle\rangle$. It should be—**is**: *Soundness*—evaluated equal to the latter, on $\langle \mathbb{X} \dot{\times} \nu\mathbb{N}\rangle \subset \mathbb{X}$.

Soundness **assertion** (•) for the present Freyd's *uniqueness* **case** recurs on $\check{\doteq}_i$, $\check{\doteq}_j$ turned into predicative equations '$\doteq$', these being already deduced, by hypothesis on $i, j < k$. Further ingredients are transitivity of '$\doteq$' and established properties of basic evaluation *ev* of map terms.

So here is the remaining—inductive—**proof,** prepared by

$$\begin{aligned}
\mathbf{T} \vdash\ & m\ \mathit{deff}\ \mathrm{dtree}_k/\langle y; \nu(n)\rangle \implies \\
& m\ \mathit{deff}\ \text{all of the following } \mathit{ev}\text{-terms and} \\
& \mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u; y) && (\bar{0}) \\
& \qquad \text{as well as} \\
& m\ \mathit{deff}\ \text{both of the following } \mathit{ev}\text{-terms, and} \\
& \mathit{ev}(w, \langle y; \nu(s\,n)\rangle) \doteq \mathit{ev}\,(w, \langle y;\ \ulcorner s \urcorner\ \odot \nu(n)\rangle) \\
& \doteq \mathit{ev}\,(w \odot \langle\ \ulcorner \mathrm{id} \urcorner \# \ulcorner s \urcorner\ \rangle, \langle y; \nu(n)\rangle) \\
& \doteq \mathit{ev}\,(v \odot w, \langle y; \nu(n)\rangle), && (\bar{s})
\end{aligned}$$

the same being true for $w_0 := v^{\$} \odot \langle u\#\ \ulcorner \mathrm{id} \urcorner\ \rangle$ in place of w, once more by (characteristic) double recursive equations for ev, this time with respect to the *initialised internal iterated* itself.

$(\bar{0})$ and $(\bar{s})$ put together for both then show, by induction on *iteration count* $n \in \mathbb{N}$—all other free variables k, u, v, w, y together form the *passive parameter* for this induction—*truncated Soundness* assertion $(\bullet)$ for this *Freyd's uniqueness* case, namely

$$\begin{aligned}
\mathbf{T} \vdash\ & m\ \mathit{deff}\ \mathrm{dtree}_k/\langle y; \nu(n)\rangle \implies \\
& m\ \mathit{deff}\ \textit{all of the ev-terms concerned above, and} \\
& \mathit{ev}\,(w, \langle y; \nu(n)\rangle) \doteq \mathit{ev}\,(v^{\$} \odot \langle u\#\ \ulcorner \mathrm{id} \urcorner\ \rangle, \langle y; \nu(n)\rangle).
\end{aligned}$$

Induction runs as follows:

Anchor $n = 0$:

$$\mathit{ev}\,(w, \langle y; \nu(0)\rangle) \doteq \mathit{ev}\,(u, y) \doteq \mathit{ev}\,(w_0, \langle y; \nu(0)\rangle),$$

step: m *deff* etc. $\implies$

$$ev\,(w, \langle y; \nu(n)\rangle) \doteq ev\,(w_0, \langle y; \nu(n)\rangle) \implies:$$
$$ev\,(w, \langle y; \nu(s\,n)\rangle) \doteq ev\,(v, ev\,(w, \langle y; \nu(n)\rangle))$$
$$\doteq ev\,(v, ev\,(w_0, \langle y; \nu(n)\rangle)) \doteq ev\,(w_0, \langle y; \nu(s\,n)\rangle),$$

the latter since evaluation ev preserves predicative equality '$\doteq$' (Leibniz) **q.e.d.** *Termination Conditioned PR Soundness Theorem.*

Comment: Already for stating the evaluations, we needed the—categorical, free-variables theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PR}\mathbb{X}, \mathbf{PRa}\mathbb{X}$ of Primitive Recursion, as well as—for termination, even in classial frame $\mathbf{T}$—PR complexities within $\mathbb{N}[\omega]$. Since this type of **Soundness** is a corner stone in our approach, the above complicated categorical combinatorics seem to be necessary, even for the negative results on classical Foundations.

Quantified-Frame Remark on ES Theorem: set theory, $\mathbf{T}$ say, admits only finite descending chains in $\mathbb{N}[\omega]$, such $\mathbf{T}$ amits *well-order schema*

$$(\mathrm{wo}(\mathbb{N}[\omega])) \quad \frac{\alpha = \alpha(n) : \mathbb{N} \to \mathbb{N}[\omega]}{\forall n\,[\alpha(n) > 0 \Rightarrow \alpha(n+1) < \alpha(n)] \implies \exists n_0\, \alpha(n_0) = 0.}$$

For these—classical—theories $\mathbf{T}$ as frame, *Termination* of all evaluations is guaranteed, and we get in fact **Corollary** above on $\mathbf{T}$-*framed Soundness* of Theory **PRa**, for **set** theories $\mathbf{T}$, as well as already for Peano Arithmetic $\mathbf{PA}^+ = \mathbf{PA} + \mathrm{wo} = \mathbf{PA} + (\mathrm{wo}(\mathbb{N}[\omega])$ with lexicographic order on $\mathbb{N}[\omega]$ a well-order.

5.2 Framed Consistency

From **Termination-Conditioned Soundness**—resp. from **T**-framed PR Soundness—we get

π**R-framed Internal PR Consistency Corollary:** For *Descent* Theory $\pi\mathbf{R} = \mathbf{PRa} + (\pi)$, **axiom** (π) stating non-infinite iterative descent in *Ordinal* $\mathbb{N}[\omega]$, we have

$$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}}, \text{ i.e. "necessarily" in } \textit{Free-Variables} \text{ form:}$$

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to 2,\ k \in \mathbb{N} \text{ free,}$$

$$\mathbf{T} \vdash \mathrm{Con}_{\mathbf{PR}\mathbb{X}} :$$

Theory $\pi\mathbf{R}$*—as well as* ***set*** *theories* $\mathbf{T}$ *as an extension of* $\pi\mathbf{R}$—derive *that no* $k \in \mathbb{N}$ *is the internal* $\mathbf{PR}\mathbb{X}$-Proof *for* $\ulcorner \mathrm{false} \urcorner$.

Proof for this **Corollary** from *Termination-Conditioned Soundness:* By assertion (iii) of that **Theorem,** with $\chi = \chi(a) := \mathrm{false}(a) = \mathrm{false} : \mathbb{1} \to 2$, we get:

Evaluation-effective internal inconsistency of $\mathbf{PR}\mathbb{X}$—i.e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner \mathrm{false} \urcorner$—*implies* false :

$$\mathbf{PRa},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) \ \wedge\ c_d\ e_d^m(\mathrm{dtree}_k/\langle 0 \rangle) \doteq 0$$
$$\implies \mathrm{false}.$$

Contraposition to this, still with $k, m \in \mathbb{N}$ free:

$$\pi\mathbf{R} \vdash \ \mathrm{true} \implies \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) \vee c_d\ e_d^m(\mathrm{dtree}_k/\langle 0 \rangle) > 0,$$

i.e. by Free-Variables (Boolean) tautology:

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner) \implies c_d\ e_d^m(\mathrm{dtree}_k/\langle 0 \rangle) > 0 : \mathbb{N}^2 \to 2.$$

For k "fixed", the conclusion of this implication—m free—means infinite desccent in $\mathbb{N}[\omega]$ of iterative argumented deduction-tree evaluation ev_d on $\mathrm{dtree}_k/0$, which is excluded intuitively. Formally it is excluded within our Theory $\pi\mathbf{R}$ taken as frame:

We apply non-infinite-descent schema (π) to ev_d, which is given by *step* e_d and complexity c_d—the latter descends (this is *Argumented-Tree Evaluation Descent*) with each application of e_d, as long as complexity $0 \in \mathbb{N}[\omega]$ is not ("yet") reached. We combine this with—choice of—*overall "negative"* condition

$$\psi = \psi(k) := \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner\,) : \mathbb{N} \to \mathbb{2},\ k \in \mathbb{N} \text{ free}$$

and get—by that schema (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash\ \neg\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner \mathrm{false} \urcorner\,) : \mathbb{N} \to \mathbb{2},\ k \in \mathbb{N} \text{ free, i.e.}$$
$$\pi\mathbf{R} \vdash\ \mathrm{Con}_{\mathbf{PR}\mathbb{X}} \quad \textbf{q.e.d.}$$

So "slightly" strengthened Theory $\pi\mathbf{R} = \mathbf{PRa} + (\pi)$ derives FV Consistency Formula of Theory $\mathbf{PR}\mathbb{X}$ of Primitive Recursion.

Schema (π) holds in **set** Theory, since there $O := \mathbb{N}[\omega]$ is an *Ordinal,* not quite to identify with set theoretical Ordinal $\omega^\omega = \mathbb{N}[\omega]$, because classical ordinal addition on ω^ω does not commute, e.g. classically $\omega + 1 \neq 1 + \omega = \omega$. As linear *orders* (with non-infinite descent) the two are identic.

As is well known, Consistency Provability and *Soundness* of a theory are strongly tied together. We get in fact even

Theorem on πR-framed Objective Soundness of Theory PRa :

- for a **PRa** predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2}.$$

- more general, for **PRa**-maps $f, \ g : A \to B$ we have

$$\pi\mathbf{R} \vdash \ \ulcorner f\urcorner \check{\doteq}_k \ulcorner g\urcorner \implies f(a) \doteq_B g(a) : \mathbb{N} \times A \to \mathbb{2}.$$

[Same for **set** theory **T** taken as frame]

Proof of first assertion is a slight generalisation of proof of *framed Internal Consistency* above as follows—take predicate χ instead of false :

Use *Termination-Conditioned Soundness,* assertion (iii) directly:

Evaluation-effective internal Provabiliity of $\ulcorner\chi\urcorner$ within **PRa**—i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner\chi\urcorner$—*implies* $\chi(a), a \in A$ free :

$$\begin{aligned}\mathbf{PRa},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \wedge \ c_d \ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) \doteq 0 \\ \implies \chi(a) : \mathbb{N}^2 \times A \to \mathbb{2}.\end{aligned}$$

Boolean free-variables calculus, tautology

$$[\alpha \wedge \beta \Rightarrow \gamma] = [\neg\,[\alpha \Rightarrow \gamma] \Rightarrow \neg\beta]$$

(test with $\beta = 0$ as well as with $\beta = 1$),

gives from this, still with k, m, a free:

$$\begin{aligned}\pi\mathbf{R} \vdash \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \Rightarrow \chi(a)\,] \\ \implies c_d \ e_d^m(\mathrm{dtree}_k/\langle 0\rangle) > 0 : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{2}.\end{aligned}$$

As before, we apply non-infinit schema (π) to ev_d, in combination with—choice of—*overall "negative"* condition

$$\psi = \psi(k, a) := \neg\,[\,\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner\,) \,\Rightarrow\, \chi(a)\,] : \mathbb{N} \times A \to \mathbb{2},$$

and get—schema (π)—overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash\; \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner\,) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2}.$$

q.e.d. for first assertion.

For **proof** of second assertion, take in the above

$$\chi = \chi(a) := [\,f(a) \doteq g(a)\,] : A \to B^2 \to \mathbb{2}$$

and get

$$\begin{aligned}\pi\mathbf{R} \vdash\; & \ulcorner f\urcorner \check{\doteq}_k \ulcorner g\urcorner \\ & \implies \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(j(k), \ulcorner f \doteq g\urcorner\,) \\ & \quad\;\; \text{(substitutivity into } \doteq) \\ & \implies [\,f(a) \doteq g(a)\,] : \mathbb{N} \times A \to \mathbb{2} \quad \textbf{q.e.d.}\end{aligned}$$

5.3 Decidability of PR Predicates

As the kernel of decision for PR predicate $\chi = \chi(a) : A \to \mathbb{2}$ *by* Theory $\mathbf{S}$ strengthening $\pi\mathbf{R}$, in particular $\mathbf{S} = \pi\mathbf{R}$ or $\mathbf{S}$ a **set** theory, we introduce a (partially defined) μ-recursive *decision algorithm* $\nabla\chi = \nabla^{\mathrm{PR}}\chi : \mathbb{1} \rightharpoonup \mathbb{2}$ for (individual) χ. This decision algorithm is viewed as a map of Theory $\widehat{\mathbf{S}}$, of *partial* $\mathbf{S}$ maps.

As a *partial* PR map it is given—see chapter 2—by three (PR) data:

- its index domain $D = D_{\nabla\chi}$, typically (and here): $D \subseteq \mathbb{N}$,
- its enumeration $d = d_{\nabla\chi} : D \to \mathbb{1}$ of its *defined arguments,* as well as
- its *rule* $\widehat{\nabla} = \widehat{\nabla}\chi : D \to \mathbb{2}$ mapping indices k, k' in D pointing to the same argument $d(k) \doteq d(k')$ in Domain $\mathbb{1}$, to the same *value* $\widehat{\nabla}(k) \doteq \widehat{\nabla}(k')$.

Now **define** alleged decision algorithm by fixing its *graph*

$$\nabla\chi = \langle (d, \widehat{\nabla}) : D \to \mathbb{1} \times \mathbb{2} \rangle : \mathbb{1} \rightharpoonup \mathbb{2}$$

as follows:

Enumeration *Domain for defined arguments* is to be

$$D = D_{\nabla\chi} \ =_{\mathrm{def}} \ \{k \,|\, \neg\,\chi\,\mathrm{ct}_A(k) \vee \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner)\} \subset \mathbb{N},$$

with $\mathrm{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count, A assumed pointed.

Defined arguments *enumeration* is here “simply”

$$d \ =_{\mathrm{def}} \ \Pi : D \xrightarrow{\subseteq} \mathbb{N} \xrightarrow{\Pi} \mathbb{1}$$

—not a priori a retraction or empty—, and *rule* is taken

$$\widehat{\nabla}(k) = \widehat{\nabla}\chi(k) \ =_{\mathrm{def}} \ \begin{cases} \text{false if } \neg\,\chi\,\mathrm{ct}_A(k), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \end{cases} : D_{\nabla\chi} \to \mathbb{2}.$$

$\widehat{\nabla} : D \to \mathbb{2}$ is in fact a well defined *rule* for *enumeration* $d : D \to \mathbb{N} \to \mathbb{1}$ of *defined argument(s)* since by (earlier) *Framed Logical Soundness Theorem*

$$\mathbf{S} \vdash \ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

whence disjointness of the alternative within $D = D_{\nabla\chi}$, and therefore we get in fact right uniqueness of alleged partial PR map $\nabla : \mathbb{1} \rightharpoonup \mathbb{2}$, namely

$$\begin{aligned}
&[d(k) \doteq d(k')] \; [\doteq \text{true since } \mathbb{1} \text{ terminal}] \\
&\implies \widehat{\nabla}(k) \doteq \widehat{\nabla}(k') \doteq \text{true} \,\dot{\vee}\, \widehat{\nabla}(k) \doteq \widehat{\nabla}(k') \doteq \text{false} \\
&\implies \widehat{\nabla}(k) \doteq \widehat{\nabla}(k') \text{ anyway.}
\end{aligned}$$

This taken together means intuitively (within $\pi\mathbf{R}$)—and formally within **set** theory $\mathbf{T}$:

$$\nabla(k) = \nabla\chi(k) = \begin{cases} \text{false if } \neg\,\chi\,\mathrm{ct}_A(k), \\ \text{true if } \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner), \\ \textit{undefined otherwise.} \end{cases}$$

We have the following complete—metamathematical—**case distinction** on $D_{\nabla\chi} \subset \mathbb{N}$:

- **1st case,** termination: D has at least one ("total"), PR point $\mathbb{1} \to D \subseteq \mathbb{N}$, and hence

  $$t = t_{\nabla\chi} \; =_{\text{by def}} \; \mu D : \mathbb{1} \to D$$

  is a (total) PR point.

  Subcases:

  – **1.1st,** negative (total) **subcase:**
  $\neg\,\chi\,\mathrm{ct}_A(t) = \text{true}$. [**Then** $\nabla\chi = \text{false}$.]

- **1.2nd,** positive (total) **subcase:**

  $\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) = \mathrm{true}$. [**Then** $\nabla\chi = \mathrm{true}$.]

  These two **subcases** are ***disjoint,*** disjoint here by π**R framed Soundness** of Theory **PR**$\mathbb{X}$ which reads (cf. above)

  $$\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \\ \mathbb{N} \times A \to \mathbb{2},\ k \in \mathbb{N} \text{ free, and } a \in A \text{ free,}$$

  here in particular—substitute $t : \mathbb{1} \to \mathbb{N}$ into k free:

  $$\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) \implies \chi(a) : A \to \mathbb{2},\ a \text{ free.}$$

  So furthermore, by this framed Soundness, in present **subcase:**

  $$\mathbf{S} \vdash \chi(a) \wedge \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(t, \ulcorner\chi\urcorner) : A \to \mathbb{2}.$$

- **2nd case,** derived non-termination:

  $\mathbf{S} \vdash D = \emptyset_{\mathbb{N}} \equiv \{\mathbb{N} \,|\, \mathrm{false}_{\mathbb{N}}\} \subset \mathbb{N}$

  [then in particular $\mathbf{S} \vdash \neg\chi = \mathrm{false}_A : A \to \mathbb{2}$,

  so $\mathbf{S} \vdash \chi$ in this case],

  and

  $\mathbf{S} \vdash \neg\mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k, \ulcorner\chi\urcorner) : \mathbb{N} \to \mathbb{2}$, k free;

- **3rd, remaining,** *ill* **case** is:

  <u>for each</u> PR point $p : \mathbb{1} \to D$ $\mathbf{S} \nvdash D(p) = \mathrm{true}$ as well as $\mathbf{S} \nvdash D = \emptyset\, [\equiv \mathrm{false}_{\mathbb{N}}]$:

  D (metamathematically) *has no (total) points* $\mathbb{1} \to D$, *but is nevertheless not empty.*

Taking in the above the **(disjoint) union** of **2nd subcase** of **1st case** and of **2nd case,** and formalising last, Remaining case for quantified frame $\mathbf{T}$ as well as for frame $\pi\mathbf{R}$ we **arrive at** the following

Quasi-Decidability Theorem: PR predicates $\chi : A \to 2$ give rise within Theory $\mathbf{S} = \mathbf{T}$ resp. $\pi\mathbf{R}$ to the following **complete (metamathematical) case distinction:**

(a) $\mathbf{S} \vdash \chi : A \to 2$ **or else**

(b) $\mathbf{S} \vdash \neg\,\chi\,\mathrm{ct}_A\,t : \mathbb{1} \to D_{\nabla\chi} \to 2$

(*defined counterexample*), **or else**

(c) $D = D_{\nabla\chi}$ ***irrefutably*** *non-empty, pointless,* formally: in this **case** we would have ***irrefutably*** over $\mathbf{S}$:

$\exists\,\hat{a} \in D,$ resp. for $\mathbf{S}$ in general:

$[\mu D \in D] \;=_{\text{by def}}\; [D\,\widehat{\circ}\,\mu D \mathrel{\widehat{=}} \text{true} : \mathbb{1} \rightharpoonup \mathbb{N} \to 2]$

<u>and</u>

$\neg\, D \circ p = \text{true} : \mathbb{1} \to \mathbb{N} \to 2$ <u>for each</u> PR point $p : \mathbb{1} \to \mathbb{N}$.

We **rule out** the latter—general—possibility of a *non-empty, pointless* predicate, for quantified arithmetical frame theory $\mathbf{T}$ by gödelian ***assumption*** of ω-consistency which rules out above instance of *ω-inconsistency.*

For frame $\pi\mathbf{R}$ we rule it out by (corresponding) metamathematical ***assumption*** of "μ-consistency," as follows:

Intermission on two variants of ω-consistency:

Gödelian ***assumption*** of ω-consistency—non-ω-inconsistency—for a *quantified* arithmetical theory $\mathbf{T}$ reads:

For **no** PR predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\mathbf{T} \vdash \ (\exists\, n \in \mathbb{N})\, \varphi(n)$$

and (nevertheless)

$$\mathbf{T} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \neg\, \varphi(2),\ \ldots$$

Adaptation to (categorical) **recursive** Theory $\pi\mathbf{R}$ is the following ***assumption*** of μ-consistency, non-μ-inconsistency for $\pi\mathbf{R}$:

For **no** PR predicate $\varphi : \mathbb{N} \to \mathbb{2}$

$$\pi\mathbf{R} \vdash \ \varphi(\mu\varphi) \ =_{\text{by def}} \ \varphi \,\widehat{\circ}\, \mu\varphi \,\widehat{=}\, \text{true} : \mathbb{1} \rightharpoonup \mathbb{2}$$

and

$$\pi\mathbf{R} \vdash \neg\, \varphi(0),\ \neg\, \varphi(1),\ \ldots\ ,\ \neg\, \varphi(\text{num}(\underline{n})),\ \ldots$$

For quantified $\mathbf{T}$ first line reads: $\mathbf{T} \vdash \exists n\, \varphi(n)$, and hence μ-consistency is equivalent to gödelian ω-consistency for such $\mathbf{T}$.

Alternative to μ-consistency: π-consistency.

By assertion (iii) of **Structure Theorem** in chapter 2—*section lemma*—for Theories $\widehat{\mathbf{S}}$ of partial PR maps, first factor $\mu\varphi : \mathbb{1} \rightharpoonup \mathbb{N}$ of (total) PR map true : $\mathbb{1} \to \mathbb{2}$ above is necessarily itself a—*totally defined*—PR map: Intuitively, a first factor of a total map cannot have undefined arguments, since these would be undefined for the composition.

Now consider—here available—(external) point <u>evaluation</u> into numerals[4], externalisation of Objective evaluation

$$ev : \lceil \mathbb{1}, \mathbb{N} \rceil \xrightarrow{\cong} \lceil \mathbb{1}, \mathbb{N} \rceil \times \mathbb{1} \xrightharpoonup{ev} \mathbb{N} \xrightarrow{\cong} \nu\mathbb{N} \subseteq \lceil \mathbb{1}, \mathbb{N} \rceil$$

[4] LASSMANN 1981

of point codes into (internal) numerals, $ev(u) \check{=} u \in \lceil \mathbb{1}, \mathbb{N} \rceil$.

This externalised evaluation $\underline{ev}$ is ***assumed***—meta-**axiom** of π-consistency—to (correctly) terminate:

$$\pi\mathbf{R}(\mathbb{1}, \mathbb{N}) \supset \mathrm{num}\, \underline{\mathbb{N}} \ni \underline{ev}(p) =^{\pi} p \in \pi\mathbf{R}(\mathbb{1}, \mathbb{N}).$$

Comment: π-consistency means *Semantical Completeness* of Descent axiom (π), this axiom is modeled into the external world of PR Metamathematic. But π-consistency is somewhat stronger: it assumes termination of $\underline{ev}$ instead of non-infinite descent.

Non-μ-inconsistency (of $\pi\mathbf{R}$) is then a consequence of π-consistency of Theory $\pi\mathbf{R}$ above:

$$\begin{aligned} &\pi\mathbf{R} \vdash \ \mathrm{true} = \varphi(\mu\varphi) = \varphi \,\hat{\circ}\, \mu\varphi = \varphi \circ \mu\varphi : \mathbb{1} \to \mathbb{N} \to \mathbb{2} \\ &\text{entails } \pi\mathbf{R} \vdash \ \neg\,(\neg\,\varphi(\mathrm{num}(\underline{n}_0))), \text{with } \ \underline{ev}(\mu\varphi) = \mathrm{num}(\underline{n}_0). \end{aligned}$$

End of Intermission.

First **Consequence:** Theory $\pi\mathbf{R}$ admits **no** non-empty predicative subset $\{n \in \mathbb{N} \mid \varphi(n)\} \subseteq \mathbb{N}$ such that for each numeral $\mathrm{num}(\underline{n}) :$ $\mathbb{1} \to \mathbb{N}$

$$\pi\mathbf{R} \vdash \neg\,\varphi \circ \mathrm{num}(\underline{n}) : \mathbb{1} \to \mathbb{N} \to \mathbb{2}.$$

This rules out—in *Quasi-Decidability* above—possibility (c) for decision Domain $D = D_{\nabla_\chi} \subseteq \mathbb{N}$ of decision operator ∇_χ for predicate $\chi : A \to \mathbb{2}$, and we get two unexpected results:

Decidability Theorem: Each free-variable PR predicate $\chi :$ $A \to \mathbb{2}$ gives rise to the following **complete case distinctions:**

- Under ***assumption*** of μ-consistency or π-consistency for $\pi\mathbf{R}$:

- $\pi\mathbf{R} \vdash \ \chi(a) : A \to \mathbb{2}$ (*theorem*) **or**
- $\pi\mathbf{R} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla\chi} \to \mathbb{2}$
  (*defined counterexample.*)

• Under ***assumption*** of ω-consistency for **set** theory $\mathbf{T}$:

- $\mathbf{T} \vdash \ \chi(a) : A \to \mathbb{2}$ (*theorem*) **or**
- $\mathbf{T} \vdash \neg\,\chi\,\mathrm{ct}_A\,\mu D : \mathbb{1} \to D_{\nabla\chi} \to \mathbb{2}$, i. e.
  $\mathbf{T} \vdash (\exists\, a \in A)\,\neg\,\chi(a)$.

Take here, in case of **set** theory $\mathbf{T}$, for predicate χ, $\mathbf{T}$'s own free-variable consistency formula $\mathrm{Con}_{\mathbf{T}} = \neg\,\mathrm{Prov}_{\mathbf{T}}(k,\ \ulcorner \mathrm{false} \urcorner\ ) : \mathbb{N} \to \mathbb{2}$, and get, under ***assumption*** of ω-consistency for $\mathbf{T}$, **consistency decidability** for $\mathbf{T}$.

This contradiction to (the postcedent) of Gödel's **2nd Incompleteness Theorem** shows that the ***assumption*** of ω-completeness for **set** theories $\mathbf{T}$ must fail.

Now take in the Theorem for χ $\pi\mathbf{R}$'s own free variable PR consistency formula

$$\mathrm{Con}_{\pi\mathbf{R}} = \neg\,\mathrm{Prov}_{\pi\mathbf{R}}(k,\ \ulcorner \mathrm{false} \urcorner\ ) : \mathbb{N} \to \mathbb{2} \text{ and get}$$

Consistency Decidability for Descent Theory $\pi\mathbf{R}$:

• $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}} : \mathbb{1} \to \mathbb{2}$ **or else**

• $\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_{\pi\mathbf{R}}$, will say
  $\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(\mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k,\ \ulcorner \mathrm{false} \urcorner\ ),\ \ulcorner \mathrm{false} \urcorner\ ) = \mathrm{true}$ **q.e.d.**

Consistency Provability Theorem: $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$, under ***assumption*** of π-consistency of Theory $\pi\mathbf{R}$.

Proof: Suppose we have 2nd alternative in *Consistency Decidability* above,

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(t, \ulcorner \mathrm{false} \urcorner),$$

$t =_{\mathrm{def}} \mu\,\mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{1} \to \mathbb{N}$, necessarily (”total”) PR. Meta PR point evaluation $\underline{ev}$ would turn—π-consistency—t into a numeral $\mathrm{num}(\underline{k}_0) : \mathbb{1} \to \mathbb{N}$, $\underline{k}_0 \in \underline{\mathbb{N}}$, $\mathrm{num}(\underline{k}_0) =^{\pi} t$, hence

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}_0), \ulcorner \mathrm{false} \urcorner).$$

But by derivation-into-*Proof* internalisation we have

$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mathrm{num}(\underline{k}), \ulcorner \chi \urcorner)$ (only) iff $\pi\mathbf{R} \vdash_{\underline{k}} \chi$, whence we would get inconsistency $\pi\mathbf{R} \vdash_{\underline{k}_0}$ false. This rules out in fact 2nd alternative in Consistency Decidability and so proves the **Theorem.**

For Proof of *Soundness* of $\pi\mathbf{R}$ below we need

ν-Lemma for Theory $\pi\mathbf{R}$:

(i) family $\nu_A : A \to [\mathbb{1}, A]_\pi = \lceil \mathbb{1}, A \rceil / \check{=}^{\pi}$ is a natural transformation, will say

$$\begin{aligned} &(\nu_B \circ f)(a) = \nu_B(f(a)) \\ &\check{=}^{\pi}_{k(a)} \ulcorner f \urcorner \odot \nu_A(a) && (*) \\ &= [\mathbb{1}, f]_\pi(\nu_A(a)), \\ &k(a) : A \to \mathbb{N} \text{ suitable PR.} \end{aligned}$$

As a commuting DIAGRAM:

$$\begin{tikzcd}
A \ni a \arrow[r, mapsto, "\nu_A"] \arrow[dd, "f"] & \nu_A(a) \in \lceil \mathbb{1}, A \rceil \arrow[d, "{\lceil \mathbb{1}, f \rceil}"] \\
 & \ulcorner f \urcorner \odot \nu_A(a) \arrow[d, phantom, "\check{\doteq}^{\pi}"] \\
B \ni f(a) \arrow[r, mapsto, "\nu_B"] & \nu_B f(a) \in \lceil \mathbb{1}, B \rceil
\end{tikzcd}$$

(ii) $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$ is injective, i. e.

$$\nu(m) \check{\doteq}^{\pi} \nu(n) \implies m \doteq n.$$

(iii) same for all Objects A of $\pi\mathbf{R}$: $\nu_A = \nu_A(a) : A \to [\mathbb{1}, A]_\pi$ is injective.

Proof: We show assertion (i) by structural recursion on $f : A \to B$.

anchor cases $f = \mathrm{id}_A$ as well as $f = 0 : \mathbb{1} \to \mathbb{N}$ are obvious.

anchor case $f = s : \mathbb{N} \to \mathbb{N}$:

$\nu(s(a)) \ =_{\text{by def}} \ \ulcorner s \urcorner \odot \nu(a) = [\mathbb{1}, s]\,(\nu(a)).$

Map composition $g \circ f : A \to B \to C$: combine the two commuting squares for f and for g into commuting rectangle for $g \circ f$.

Cartesian Structure: use

$\nu_{(A \times B)} \ =_{\text{by def}} \ \mathrm{ind} \circ (\nu_A \times \nu_B)$:

$A \times B \to \lceil \mathbb{1}, A \rceil \times \lceil \mathbb{1}, B \rceil \xrightarrow{\cong} \lceil \mathbb{1}, A \times B \rceil \to [\mathbb{1}, A \times B],$

componentwise definition of (any) equality on Cartesian product, as well as the universal properties of the Cartesian product $A \times B$ and $[\mathbb{1}, A \times B] \cong [\mathbb{1}, A] \times [\mathbb{1}, B]$, projections $[\mathbb{1}, \ell], [\mathbb{1}, r]$.

Iterated $f^{\S}(a,n) : A \times \mathbb{N} \to A$ of (already tested) endo $f : A \to A$: Straight forward by recursion on n, since iteration is repeated composition.

Assertion (ii) on injectivity of $\nu = \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$:

$$\begin{aligned}
&\nu(m) \,\check{=}^{\pi}\, \nu(n) \implies \ulcorner \dot{=} \urcorner \odot (\nu(m) \times \nu(n)) \,\check{=}^{\pi}\, \ulcorner \mathrm{true} \urcorner \\
&\qquad \text{by internal substitutivity into predicative equality } \dot{=} \\
&\iff \lceil \mathbb{1}, \dot{=} \rceil \circ (\nu \times \nu)(m,n) \,\check{=}^{\pi}\, \ulcorner \mathrm{true} \urcorner \\
&\implies \nu_{\mathbb{2}}[m \dot{=} n] \,\check{=}^{\pi}\, \nu_{\mathbb{2}}(\mathrm{true}) \\
&\qquad \text{by naturality of transformation } \nu \\
&\implies m \dot{=} n, \text{ by } \textit{self-consistency} \text{ (!) of Theory } \pi\mathbf{R}.
\end{aligned}$$

General ν injectivity assertion (iii) now follows from that special just above, from componentwise definition of ν—and componentwise definition of injectivity—on Cartesian products (and restriction of both to predicative subObjects), via naturality of transformation $[\, \nu_A : A \to [\mathbb{1}, A]_\pi \,]_{A \in \pi\mathbf{R}}$ **q.e.d.**

This is to give self-consistency $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ to be **equivalent** to

Objective Soundness Theorem for Descent Theory $\pi\mathbf{R}$:

- for $\pi\mathbf{R}$-maps $f,\ g : A \to B$:

$$\pi\mathbf{R} \vdash [\ \ulcorner f \urcorner \,\check{=}^{\pi}_{k}\, \ulcorner g \urcorner\ ] \implies f(a) \dot{=}_B g(a) : \mathbb{N} \times A \to \mathbb{2}.$$

- this gives in particular *Logical Soundness* of Theory $\pi\mathbf{R}$:

  For a predicate $\chi = \chi(a) : A \to \mathbb{2}$ we have

$$\pi\mathbf{R} \vdash \ \mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner\chi\urcorner) \implies \chi(a) : \mathbb{N} \times A \to \mathbb{2},$$

  $a \in A$ free, meaning here $\forall a$, and $k \in \mathbb{N}$ free, meaning here $\exists k$.

Proof: Granted Self-Consistency of Theory $\pi\mathbf{R}$ means just injectivity of numeralisation

$$\nu_{\mathbb{2}} : \mathbb{2} \to [\mathbb{1}, \mathbb{2}]_\pi = \lceil\mathbb{1}, \mathbb{2}\rceil / \check{\doteq}^\pi.$$

The **Lemma** deduces that this injectivity carries over first to numeralisation $\nu_{\mathbb{N}} = \nu : \mathbb{N} \to [\mathbb{1}, \mathbb{N}]_\pi$, and then to all numeralisations

$$\nu_B : B \to [\mathbb{1}, B]_\pi, \ \ B \text{ a } \pi\mathbf{R} \text{ Object}.$$

Now compatibility of internal composition with internal equality as well as—**Lemma** again—naturality of transformmation $\nu_A : A \to [\mathbb{1}, A]_\pi$ give

$$\begin{aligned} \pi\mathbf{R} \vdash [\ \ulcorner f\urcorner \ \check{\doteq}^\pi_k \ \ulcorner g\urcorner\ ] & \\ \implies & \ulcorner f\urcorner \odot \nu_A(a) \,\check{\doteq}^\pi\, \ulcorner g\urcorner \odot \nu_A(a) \\ \implies & \nu_B(f(a)) \,\check{\doteq}^\pi\, \nu_B(g(a)) \\ \implies & f(a) \doteq g(a), \end{aligned}$$

the latter implication following from injectivity of $\nu_B : B \to [\mathbb{1}, B]_\pi$ **q.e.d.**

ω-**Completeness Theorem** for Theory $\pi\mathbf{R}$: Theory $\pi\mathbf{R}$ admits the following schema of *test by all internal numerals:*

$$(\omega\text{−Comp})\quad \frac{\begin{array}{l}\chi = \chi(a) : A \to 2 \text{ predicate,}\\ k = k(a) : A \to \mathbb{N} \text{ such that}\\ \pi\mathbf{R} \vdash\ \mathrm{Prov}_{\pi\mathbf{R}}(k(a),\ \ulcorner\chi\urcorner\ \odot \nu_A(a)) : A \to 2\end{array}}{\pi\mathbf{R} \vdash\ \chi : A \to 2.}$$

Proof: By ν naturality—within $\pi\mathbf{R}$—the antecedent gives

$$\pi\mathbf{R} \vdash\ \mathrm{Prov}_{\mathbf{PR}\mathbb{X}}(k'(a), \nu_2 \circ \chi(a)) : A \to 2,$$

and from this, by $\pi\mathbf{R}$ self-consistency: injectivity of ν_2 within $\pi\mathbf{R}$,

$$\pi\mathbf{R} \vdash \chi(a) : A \to 2 \quad \textbf{q.e.d.}$$

Interpretation: The $\nu_A(a), a \in A$ are jointly epic, νA lies *dense* in $[\mathbb{1}, A]_\pi$. Theory $\pi\mathbf{R}$ is in particular internally μ-consistent, Object $\mathbb{1}$ is an internal separator, all of this with respect to $\pi\mathbf{R}$ maps (on Object language level). Would it work for (free variable) internal map codes either?

Question: Can we then have/assume this test to work on the external level too? can we have/assume at least Object $\mathbb{1}$ to be/to become a *separator* for category $\pi\mathbf{R}$?

Attempt to an answer: Logic/arithmetic externalisation of **axioms** and **theorems,** as opposite to—successfull—internalisation/arithmetisation seems me to be legitimate/consistent: both internalisation and externalisation can be seen/formalised as preserving/reflecting logical *invariants.* A theory $\mathbf{T}$ for which this is not always possible—Consistency/*consistency provability*—has a defect in this regard, it is not *sound* in the technical sense, see SMORYNSKI 1977.

Conclusion: Descent Theory $\pi\mathbf{R}$—in the role of Metamathematic—derives its own *consistency* (formula) as well as—see below—the *inconsistency* (formulae) for **set** theories $\mathbf{T}$, the latter including Peano-Arithmetic $\mathbf{PA}^+$ with order of $\mathbb{N}[\omega]$ to satisfy finite descent.

All of this under ***assumption***, meta-axiom, that Theory $\pi\mathbf{R}$ is π-consistent, that it externalises its **axiom** (π) into (correct) termination of (external) evaluation $\underline{ev}$.

The $\pi\mathbf{R}$ (in part) internal version of μ-consistency, consequence of π-consistency, is ω-completeness above.

Question: Are Quantified Arithmetical Theories $\mathbf{T}$, in particular Theory $\mathbf{PA}^+$, even **inconsistent?**

By Gödel's 2nd Incompleteness Theorem, first assertion, $\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}}$ **if** $\mathbf{T}$ consistent, hence $\pi\mathbf{R} \nvdash \mathrm{Con}_{\mathbf{T}}$ if $\mathbf{T}$ consistent: this since $\mathbf{T}$ is an extension of $\pi\mathbf{R}$. But **then,** by Decidability Theorem above, for $\pi\mathbf{R}$ and PR free-variable predicate $\mathrm{Con}_{\mathbf{T}} = \neg\,\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbb{2}$,

$$\pi\mathbf{R} \vdash \neg\,\mathrm{Con}_{\mathbf{T}}, \ \ [\text{a fortiori } \ \mathbf{T} \vdash \neg\,\mathrm{Con}_{\mathbf{T}}.]$$

Now if we take as Metamathematic the external version $\underline{\mathbf{PR}}$ of fundamental Theory $\mathbf{PR}$, then the consistency questions are open.

But **if** we take as Metamathematic an external version $\underline{\pi\mathbf{R}}$ of Descent Theory $\pi\mathbf{R}$, then we get in fact consistency of PR theories $\mathbf{PR}, \mathbf{PRa}, \mathbf{PRa}\mathbb{X}$—and of Descent Theory $\pi\mathbf{R}$—as well as inconsistency of **set** theories $\mathbf{T}$.

Problems:

(1) Is axiom schema (π) redundant, $\pi\mathbf{R} \cong \mathbf{PRa}$? Certainly not, since isotonic maps from lexicographically ordered $\mathbb{N} \times \mathbb{N}, \ldots, \mathbb{N}^+ \equiv \mathbb{N}[\omega] \equiv \omega^\omega$ to $\mathbb{N}$ are not available.

(2) Can we get *internal* Soundness for Theory $\pi\mathbf{R}$ itself? Up to now we have only *Objective* Soundness: this is the one considered by mathematical Logicians. Internal Soundness (of *evaluation* versus the Object language level) is a challenging open Problem with present approach.

Résumé and Discussion

F. W. Lawvere has introduced the Natural Numbers Object into Category Theory by a first universal property of generation of sequences out of a constant and an endo map. Eilenberg & Elgot introduced iteration of endo maps, and Freyd characterised the NNO of Lawvere by availability and uniqueness of the *initialised iterated* of an endo map, this in the context of availability of internal hom. We have taken Cartesian structure and NNO $\mathbb{N}$ in this latter sense as axiomatic foundation—without (Topos theoretic) internal hom.

This foundation has "elements of sets" only as *points* $a_0 : \mathbb{1} \to A$ of *Objects,* i. e. here essentially of $\mathbb{N}$ and its Cartesian products with itself. Cum grano salis, our *free variables*—standing for projections—can be taken as (variable) "elements". Points, constants then are *defined elements.*

Goodstein's free-variable Arithmetic, a subsystem of categorical Primitive Recursive Arithmetic, is based on four simple, tricky uniqueness axioms for addition and truncated subtraction, and shows—deep—commutativity of the maximum, and by this *Equality Definability,* i. e. definability of (fundamental) equality *between* maps by *equality predicate.*

In *Extension by predicate abstraction* we have developped Primitive Recursion in direction of an equational set theory.

Partial maps are classically introduced as right unique relations, with use of quantifiers, as for definition of maps. Brinkmann & Puppe introduce relations as pairs of maps, into the domain and into the codomain. Our partial PR maps are just such relations with in addition right uniqueness, expressed map theoretically. The classical μ-recursive maps then turned out to be just partial PR maps, and could this way be introduced quantifier-free.

Absence of quantifiers makes impossible the classical proof of Gödel's Incompleteness Theorems: Joyal has proved the Gödel Theorems for his *Arithmetical Universes.* But if I remember right, these Universes come with (boolean?) quantification, and so does—intuitionistic—*Elementary Theory of Topoi* of Lawvere & Tierney, discussed by Freyd in case that the Topos comes with an NNO.

This opens—in principle—the possibility for a free-variables, quantifier free Arithmetic as the one introduced here, to be not concerned by—derived—impossibility for *consistency provability.*

Based on this Foundation, we have defined evaluation of PR map codes as a <u>C</u>omplexity <u>C</u>ontrolled <u>I</u>teration—a formally partial but intuitively terminating PR map. We extended this evaluation to *argumented deduction trees* for the Theory of Primitive Recursion.

Within Theory $\pi\mathbf{R}$ of Primitive Recursion with non-infinite Descent of CCI's (like evaluation) we were able to derive the (free-variable) $\pi\mathbf{R}$-*framed Consistency* formula for Theory **PRa** of Primitive Recursion (with predicate abstraction), as well as $\pi\mathbf{R}$-*framed Soundness* of **PRa**.

Application of this central Soundness Theorem, has lead—under the *assumption* of *μ-consistency*—to *decidability* of all free-variable PR predicates, in particular of **set** theorie's and $\pi\mathbf{R}$'s own consistency formulae, and from this—by Gödel's 2nd Incompleteness Theorem on one hand to ω-inconsistency of **set** theory.

On the other hand, *assumed* μ-consistency, variant of ω-inconsistency, ruled out self-inconsistency for Descent Theory $\pi\mathbf{R}$, with the consequence of *self-consistency* $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ for this Theory, as well as Soundness and ω-completeness.

Joseph's Question: Is there a direct way to (internal) inconsistency of Quantified Arithmetics $\mathbf{T}$, not building on the properties of Theory $\pi\mathbf{R}$?

Answer: the inconsistency result relies heavily on Gödel's incredibly deep Incompleteness Theorems for Theories $\mathbf{T}$, obtained by PR arithmetisation of "all" Metamathematics, and on (ubiquitous) availability of '$\exists$' in $\mathbf{T}$. Logical examination of Theory **PR** and **sets**-like suitable extensions $\mathbf{PRa}, \mathbf{PRa}\mathbb{X}, \pi\mathbf{R}$ seem to be appropriate for their own sake as well as for—desired—contrast in consistency to the **set** theory case.

The PR logical means developed serve for both *ω-inconsistency* of **set** theories $\mathbf{T}$ as well as for self-*consistency* of Descent Theory $\pi\mathbf{R}$.

Present work constitutes a detailed and corrected treatment of the ensemble RCF 1, in one regard erroneous RCF 2, and RCF 4 in the Bibliography. In RCF 3 I have attempted to show incompatibility of Arithmetic with (iterated) axiomatic internal hom, via code-self-evaluation.

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